Coordinates

a guest lecture by

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Today

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Last time

Given a vector $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$, the numbers $a_1, \ldots, a_n$ are its coordinates, and if you form a vector with these numbers, you get

$$[\vec{a}]_{\mathcal{B}_{\text{std}}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

...

BUT these numbers are obtained by expressing $\vec{a}$ using the standard basis $\mathcal{B}_{\text{std}} = \{\vec{e}_1, \ldots, \vec{e}_n\}$. So everything above is relative to $\mathcal{B}_{\text{std}}$.

If you use a different basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$, as opposed to the standard basis, you can express $\vec{v}$ differently, say

$$\vec{a} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n.$$

The coordinates with respect to $\mathcal{B}$ are the values $b_1, \ldots, b_n$, and they form the coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Example

$$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 - x_3 = 0 \right\} \text{ with basis } \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

What are the coordinates, with respect to $\mathcal{B}$, of the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ?$$

Does not make sense because $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \mathbb{S}$.

Warning

It only makes sense to talk about the coordinate of a vector $\vec{a}$ with respect to a basis $\mathcal{B}$, if $\vec{a} \in \mathrm{Span}\mathcal{B}$.

Example

$$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 - x_3 = 0 \right\} \text{ with basis } \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Suppose $\vec{a} = \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix}$.

This time $\vec{a} \in \mathrm{Span}\mathcal{B}$, so it makes sense to talk about its coordinates with respect to $\mathcal{B}$.

Get vector from coordinates

Say we are given a vector $\vec{a}$ and a basis $\mathcal{B}$. How do we find $[\vec{a}]_{\mathcal{B}}$?

Example

$$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 - x_3 = 0 \right\} \text{ with basis } \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Suppose $\vec{a} = \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix}$. We need to express $\vec{a}$ in terms of the vectors in $\mathcal{B}$. So we need to find numbers b_1, b_2 such that

$$b_1 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{a} = \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix}$$

Get vector from coordinates

Example (continued)

This is solved by row reducing the system:

$$\left[\begin{array}{cc|c} -2 & 1 & -5 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} b_1 = 4 \\ b_2 = 3 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad [\vec{a}]_B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Remark

In the last example, $\mathcal{B}$ is a basis of size 2 for the subspace $\mathbb{S} \subseteq \mathbb{R}^3$, so the dimension of $\mathbb{S}$ is 2, and $\mathcal{B}$ coordinate will consist of 2 numbers.

If $\mathbb{S} \subseteq \mathbb{R}^n$ has dimension k, and $\mathcal{B}$ is a basis for $\mathbb{S}$, then for any $\vec{a} \in \mathbb{S}$, the $\mathcal{B}$-coordinate vector

$$[\vec{a}]_{\mathcal{B}}$$

has k numbers in it.

In application, the total space might be $\mathbb{R}^{100}$, but if we work in a subspace $\mathbb{S}$ of small dimension, then using coordinates within $\mathbb{S}$ will simplify computation.

Example (One more example)

$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_3 = 0 \right\}$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Suppose $\vec{a} = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$. Then

$$\vec{a} = 4 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-5) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$[\vec{a}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Get vector from coordinates

Going in the reverse direction, if we are given $[\vec{a}]_{\mathcal{B}}$?

Example

$$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 - x_3 = 0 \right\} \text{ with basis } \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Suppose $[\vec{a}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Since $\mathcal{B}$ has two elements, it make sense that a $\mathcal{B}$-coordinate vector has 2 numbers in it. Then

$$\vec{a} = 2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

Change of coordinates

If I have two bases $\mathcal{B}$ and $\mathcal{C}$, and I know $[\vec{a}]_{\mathcal{B}}$, how do I find $[\vec{a}]_{\mathcal{C}}$?

Answer

You could compute $\vec{a}$ first using $[\vec{a}]_{\mathcal{B}}$, then compute $[\vec{a}]_{\mathcal{C}}$.

OR, you could just use the **change of coordinates matrix**.

Definition (Definition 5.2.13)

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ and $\mathcal{C}$ be two bases for a subspace $\mathbb{S}$ of $\mathbb{R}^n$. The **change of coordinate matrix** from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is

$$ {}_{\mathcal{C}}P_{\mathcal{B}} = [[\vec{v}_1]_{\mathcal{C}} \quad [\vec{v}_2]_{\mathcal{C}} \quad \ldots \quad [\vec{v}_k]_{\mathcal{C}}]. \in M_{k\times k}(\mathbb{R})$$

Then for any $\vec{a} \in \mathbb{S}$ we have

$$[\vec{a}]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}}[\vec{a}]_{\mathcal{B}}.$$

Remark

Here $\mathcal{C}$ is on the left, $\mathcal{B}$ is on the right because it eats a $\mathcal{B}$ coordinates on its right, and spits out a $\mathcal{C}$ coorinates.

How do I find change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$?

Answer

The definition says

$$_{\mathcal{C}}P_{\mathcal{B}} = [[\vec{v}_1]_{\mathcal{C}} \quad [\vec{v}_2]_{\mathcal{C}} \quad \dots \quad [\vec{v}_k]_{\mathcal{C}}].$$

You know how to compute each $[\vec{v}_i]_{\mathcal{C}}$ individually, so you know how to compute the whole matrix.

OR, you could compute them all at once.

Example

$$\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 - x_3 = 0 \right\}$$ with bases

${}_{C}P_{B}$

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 3 \end{bmatrix} \right\}$$

To find $\left[\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right]_{\mathcal{C}}$ and $\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right]_{\mathcal{C}}$, we need to row reduce

$$\left[\begin{array}{cc|c} -1 & -5 & -2 \\ 1 & 4 & 1 \\ 1 & 3 & 0 \end{array}\right] \quad \text{AND} \quad \left[\begin{array}{cc|c} -1 & -5 & 1 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{array}\right]$$

Example

But we could instead just row reduce

$$\left[\begin{array}{cc|cc} -1 & -5 & -2 & 1 \\ 1 & 4 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \qquad {}_C P_B = \begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|cc} 0 & -1 & -1 & 1 \\ 1 & 4 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example

$\mathbb{S} = \mathbb{R}^3$ with bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right\}, \mathcal{B}_{\text{std}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To find ${}_{\mathcal{B}_{\text{std}}}P_{\mathcal{B}}$, we need to row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 3 \end{array}\right]$$

$${}_{B_{std}}P_{B} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 0 \\ 3 & 1 & -3 \end{bmatrix}$$

Example

To find ${}_{\mathcal{B}}P_{\mathcal{B}_{\text{std}}}$, we need to row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

But this is just the process to find...

$${}_{B}P_{B_{std}} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 0 \\ 3 & 1 & 3 \end{bmatrix}^{-1} = {}_{B_{std}}P_{B}^{-1}$$

If I change the coordinates from a $\mathcal{B}$-coordinates to a $\mathcal{C}$-coordinates, then change it back, I would just retrieve the original coordinates. Therefore applying the change of coordinates matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$, then ${}_{\mathcal{B}}P_{\mathcal{C}}$, does nothing.

Theorem

If $\mathcal{B}$ and $\mathcal{C}$ are bases for a subspace $\mathbb{S}$, then

$$ {}_{B}P_{C}\,{}_{C}P_{B} = I = {}_{C}P_{B}\,{}_{B}P_{C} $$

Next time

New chapter: Chapter 6 - Eigenvectors and Diagonalization

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