PMATH 464 Algebraic geometry

AUXILIARY NOTES FOR LECTURES BY PROF. XUEMIAO CHEN
TYPESET BY JIAHUI HUANG
UNIVERSITY OF WATERLOO

Contents

This note is intended as additional material for selected lectures. You are recommended to take your own notes in lecture and do your own research. There may be concepts from other courses (Galois theory, algebraic topology, commutative algebra, etc) that you might have not seen. You are expected to learn these concepts yourself. This note will guide you through the research process if you are lost on how to begin, and you will hopefully be able to learn later materials by yourself. Feel free to contact me if you want advice on how to research for a certain topic.

Resources: Many topics can be learned quickly by reading wiki articles, stack exchange posts, etc or asking AI. If you want more credible sources, I recommend the following:

- For algebraic topology, see Hatcher's *Algebraic Topology*. Chapter 0 and 1 are important for this course.
- For commutative algebra, see Atiyah-MacDonald's *Introduction to commutative algebra*. For instance you can find an exercise that helps you work through the proof of Hilbert's Nullstellensatz.
- For algebraic geometry, see Hartshorne's *Algebraic Geometry*. This course will mainly focus on Chapter 1.

1. Week 1

1.1. **The graph of a function is a Riemannian surface.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex single-valued function. The graph of f is $\text{graph}(f) = \{(x, f(x)) | x \in \mathbb{C}\}$. For example, if $f(x) = x^2$,

Date: May 21, 2025.

then the graph is $\{(x, x^2)|x \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Consider the map

$$\pi : \text{graph}(f) \to \mathbb{C}, \quad \pi(x, f(x)) = x$$

This is a continuous bijection, so topologically the graph is just a copy of $\mathbb{C}$. This is analogous to the real case, where the parabola $\{(x, x^2)|x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ can be projected down to the x-axis which is a continuous bijection, so topologically the parabola is just a line.

Exercise 1.1. The map π is a continuous bijection (homeomorphism), but it can have more structure. Which one of the following properties does π satisfy and which does it not?

π is a ...

(1) diffeomorphism
(2) biholomorphism
(3) symplectomorphism
(4) conformal map
(5) isometry

Remark 1.2. The graph of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is always a Riemann surface. More generally, the graph of a holomorphic function $f : X \to Y$ between complex manifolds is a complex manifold. Look up Riemannian surfaces if you are interested. $\diamond$

The graph of a multi-valued function is more complicated. We will see in the next section that the graph of the complex circle $x^2 + y^2 = 1$ is a (Riemann) surface. For the function $y = x^{1/2}$, the graph is easy: it is just $\mathbb{C}$ because of the projection $(y^2, y) \mapsto y$ is a continuous bijection. Essentially this is due to the inverse function $x = y^2$ is single-valued. This is analogous to the fact that the graph of the real function $y = \pm\sqrt{x}$ is just the parabola rotated by 90 degrees, and can be projected to the y-axis.

Exercise 1.3. You may have seen in complex analysis the Riemann surface that is the graph of the complex multi-valued function $y = \log x$. Look up what this surface is and find a picture of it.

1.2. **Gluing the branch cuts of the circle $x^2 + y^2 = 1$.** Consider the "complex circle" $\{(x, y) \in \mathbb{C}^2|x^2 + y^2 = 1\}$. Since this is a subspace defined by one equation in the 2-dimensional space $\mathbb{C}^2$, we expect it to be of dimension 1. However, this is complex dimension 1, which is real dimension 2, so the "complex circle" is a surface in the real world, and we imagine it.

The set $\{(x, y) \in \mathbb{C}^2|x^2 + y^2 = 1\}$ can be also interpreted as the graph of the multi-valued function $y = (1 - x^2)^{1/2}$. This is analogous to the example that the parabola $\{(x, y) \in \mathbb{R}^2|y = x^2\}$ is the graph of the single valued function $f(x) = x^2$, where $\text{graph}(f) = \{(x, f(x))|x \in \mathbb{R}\}$.

Recall from complex analysis that the function $y = f(x) = (1 - x^2)^{1/2}$ can be interpreted as a complex single-valued function with a branch-cut. For our purpose the branch cut we choose is by removing the interval $[-1, 1]$. Consequently, we have two ways of defining f

$$f_1 : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}, f_2 : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}$$

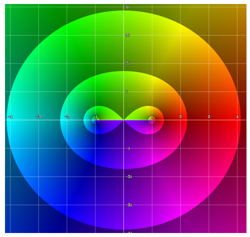
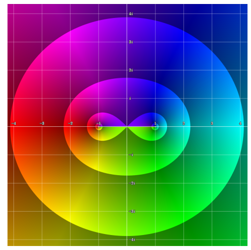
such that $f_1(x), f_2(x)$ are the two values of $(1 - x^2)^{1/2}$ for each x in their domains.

Consider the map

$$\pi_1 : \text{graph}(f_1) \to \mathbb{C} \setminus [-1, 1], \quad \pi_1(x, f_1(x)) = x.$$

This map is a continuous bijection, which means the graph of f_1 we are interested in is homeomorphic to $\mathbb{C} \setminus [-1, 1]$. You can think of this as cutting a small segment into the middle of a piece of paper with a scissor. We can write down at each point x, the value of $f_1(x)$ to keep track of the function f_1, but topologically it is a flat piece of paper. Similarly, the graph of f_2 is also homeomorphic to $\mathbb{C} \setminus [-1, 1]$.

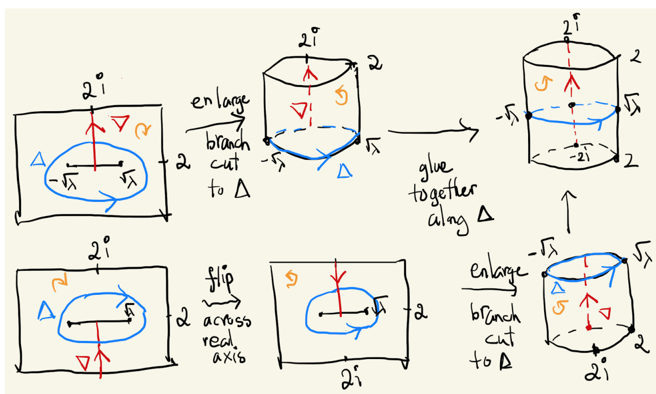
You have seen in class the following two images for a visualization of these graphs, which can be thought of as coloring the paper based on the value of f_1 and f_2.

Let us refer to the pictures. In image 1, the point 1.1 is mapped to red, while in the second image the point -1.1 is mapped to red, which reflects the fact that $f_1(1.1) = f_2(-1.1) = \textit{red}$.

Now we need to combine the two graphs into a single surface to get the (multi-valued) graph $\{(x, y) \in \mathbb{C}^2 | x^2 + y^2 = 1\}$. This is done by glueing the two pieces of papers together along the interval $[-1, 1]$. Note that when the paper is cut, the interval $[-1, 1]$ splits into two, one attached to the top half of the paper, and one to the bottom. In the first image, the top half is green and the bottom half blue, and vice-versa for the second image. When gluing them together, we match the colors and obtain a single surface which looks like a tube.

Exercise 1.4. I found the following image online for demonstration. Try researching yourself and see if you can find more detailed explanations of this process.



More precisely, let $\overline{\text{graph}(f_1)}$ be the topological space obtained by attaching the two points $\{\pm 1\}$ and two copies of $(-1, 1)$ (forming top and bottom copies of $[-1, 1]$), and similarly $\overline{\text{graph}(f_2)}$. The graph of the complex circle is

$$\left(\overline{\text{graph}(f_1)} \bigsqcup \overline{\text{graph}(f_2)} \right) / \sim$$

where $\sqcup$ denotes disjoint union, and $\sim$ is the equivalence relation that identifies the copies of $[-1, 1]$ by color.

Exercise 1.5. Make the above explanation completely rigorous (using algebra instead of pictures and colors). Recall the definition of quotient space and gluing from algebraic topology.

1.3. **The monodromy group of the multi-valued function $y = x^{1/2}$.**

There are many ways of observing the monodromy of a covering map, for which you may refer to your own notes from the lecture. Here we give some detail for the definition using "curve lifting".

The most common definition for *monodromy* you will find in the context of covering maps. Let X, Y be connected topological spaces. Suppose $p : X \to Y$ is a covering map of degree d for some positive integer d. We know from definition the fiber $F = p^{-1}(y)$ consists of d points for any

$y \in Y$. Also, as a property of covering maps, we have the *unique lifting property* which says: if $\gamma : [0,1] \to Y$ is a path starting at $\gamma(0)$ and ending at $\gamma(1)$, then we may specify a starting point $x \in p^{-1}(\gamma(0))$, and there exists a unique lift $\tilde{\gamma} : [0,1] \to X$ that starts at x and ends at some point in $p^{-1}(\gamma(1))$.

Exercise 1.6. Recall the definition of a covering space and the proof of unique lifting property from algebraic topology.

To define a monodromy, we also need the notion of *fundamental groups* $\pi_1(X)$. As explained in class, this can be naively understood as the group of loops on X, up to *homotopy*, i.e. two loops are considered equivalent if you can interpolate them from one to the other via a continuous family of other loops. The addition operator of the fundamental group is given by concatenating two loops.

For example, if X is $\mathbb{C} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\}$, then the fundamental group $\pi_1(X)$ is $\mathbb{Z}$, where $1 \in \mathbb{Z}$ corresponds to a loop that wraps the origin once counterclockwise, and -1 corresponds to a loop that wraps around the origin once clockwise. If a loop does not wrap around the origin, then you can shrink it to a single point (the constant loop) and say this loop is homotopic to the constant loop, which corresponds to the identity $0 \in \mathbb{Z}$.

If γ is the loop $\gamma : [0,1] \to \mathcal{C} \setminus \{0\}$ given by the circle $\gamma(t) = e^{2\pi i t}$, then the equivalence class of γ in the fundamental group, denoted $[\gamma]$, correspond to the number 1 in $\mathbb{Z}$. If σ is the double speed of γ, given by $\sigma(t) = e^{4\pi i t}$, which wraps around the origin two times, then $[\sigma]$ corresponds to 2.

Exercise 1.7. Recall the definition of fundamental group from algebraic topology. What is the fundamental group of $\mathbb{R} \setminus \{\pm 1\}$? Also take note that I have simplified the notations by assuming X, Y are connected.

Definition 1.8. The *monodromy* of the covering map $p : X \to Y$ is the group homomorphism

$$\begin{aligned} \varphi : \pi_1(Y) \to & \operatorname{Aut}(F) \cong S_d \\ [\gamma] \mapsto & (\rho : F \to F, \rho(x) = \tilde{\gamma}_x(1)) \end{aligned}$$

where $F = p^{-1}(y)$ for any $y \in Y$, S_d denotes the symmetric group of degree d, and $\tilde{\gamma}_x$ is the unique lift of γ starting at x.

The automorphism $\varphi([\gamma])$, which is a permutation on a set with d elements, is called the *monodromy action* associated to γ.

The *monodromy group* of the covering map $p : X \to Y$ is the image of φ, which is a subgroup of the symmetric group S_d.

Exercise 1.9. Convince yourself why the above definition is independent of the choice of y.

In complex analysis, the function $x^{1/2}$ only makes sense as a complex, single-valued function after we choose a branch cut on $\mathbb{C}$. However, we can also to interpret it as "multi-valued function" $y = x^{1/2}$, sending each $x \in \mathbb{C}$ to its two square roots.

Exercise 1.10. Look up a rigorous definition for "multi-valued function".

Now we want to make sense of the phrase "monodromy group of the multi-valued function $y = x^{1/2}$". But we only know what a monodromy is for covering maps. Therefore we should try to construct a covering map out of $y = x^{1/2}$, then consider the monodromy of that covering map. The map we want is

$$f : \mathbb{C} \to \mathbb{C}, \quad x \mapsto x^2,$$

which is the inverse of the function $y = x^{1/2}$. Since $x^{1/2}$ sends one point to two values, the inverse sends two point to the same value, making f a covering map of degree 2. Of course there is the exceptional point 0, over which the function is one to one. Hence we need to take out the origin, and obtain a genuine covering map

$$p := f|_{\mathbb{C}\setminus\{0\}} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}.$$

Exercise 1.11. The circle $\gamma(t) = e^{2\pi i t}$ lifts via the covering map p to a path $\tilde{\gamma}$ which starts at the point 1. What is the endpoint $\tilde{\gamma}(1)$ of this path? What about the lift whose starting point is -1?

Exercise 1.12. Based on the above exercise, what is the monodromy group of p? Remember it should be a subgroup of S_2. What is the monodromy action of the circle γ?

Now it makes sense to talk about the monodromy group of $y = x^{1/2}$: it is simply the monodromy of p.

Finally, for your assignment, you will be computing the monodromy group of the multi-valued function $x^2+y^2 = 1$ (or some other more complicated function). If you do not know how to proceed, try to first define the covering map p. The might p might not be directly analogous to the definition above, but that is what you need to figure out. Alternatively, you may use other examples such as braid groups to come up with your own definition of the monodromy. It will work as long as you have the correct understanding.

2. Week 2

This week we are looking at elliptic curves. There are many ways to define elliptic curves, and you will find out how the definitions relate to each other.

You will need to be familiar with the quotient of a topological space X by an equivalence relation $\sim$. Look it up if you have not seen this.

2.1. The torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

The notation $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is in reference to the quotient group of $\mathbb{C}$ by the subgroup $\mathbb{Z} + i\mathbb{Z}$. For an equivalence class $\overline{x + iy + (\mathbb{Z} + i\mathbb{Z})}$, we can add integers to the real and imaginary part and obtain a representative $\overline{x' + iy'}$ such that $0 \leq x' < 1$ and $0 \leq y < 1$. Pictorially this looks like the unit square. Topologically we are glueing the unit square first along its horizontal sides then along the vertical sides, which is why it is called a torus.

Remark 2.1. The quotient $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ is a group, but also a topological space, manifold, complex manifold, and in particular, an elliptic curve. $\diamond$

More generally, a lattice Γ in $\mathbb{R}^n$ is the subgroup generated by a basis of $\mathbb{R}^n$. Let Γ be a lattice in $\mathbb{C} \cong \mathbb{R}^2$. Then again the elements of the quotient group $\mathbb{C}/\Gamma$ are represented by elements of certain parallelograms. Topologically it is obtained by glueing opposite sides of the parallelogram, which makes it a torus.

Exercise 2.2. For $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, the unit square is called the fundamental parallelogram. Find out what *fundamental parallelogram* means for a general lattice $\Gamma \subseteq \mathbb{R}^n$.

2.2. The projective space.

You have seen in class what $\mathbb{P}^1_{\mathbb{R}}$ is. Here you will see the definition of $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{C}}$, which is completely analogous. In your homework you will define the *projective space* $\mathbb{P}^n_k$ for any field k and natural number n.

In words, the projective line $\mathbb{P}^1$ is the space of lines in $\mathbb{C}^2$. Alternatively, it can be defined as a quotient space.

Definition 2.3. The *projective line* $\mathbb{P}^1$ is given by

$$(\mathbb{C}^2 \setminus \{0\})/\sim$$

where $\sim$ is the equivalence relation

$$(x, y) \sim (x', y') \iff (x, y) = (kx', ky') \text{ for some } k \in \mathbb{C}.$$

It is called the projective line because it has complex dimension 1, hence the superscript 1 on $\mathbb{P}$.

Exercise 2.4 (Another definition)**.** Consider the set of lines in $\mathbb{C}^2$

$$\{L : L \subseteq \mathbb{C}^2 \text{ is a linear subspace of dimension } 1\}.$$

Find a topology on this set that makes it homeomorphic to $\mathbb{P}^1$. This will justify the above definition with the claim that $\mathbb{P}^1$ is the space of lines.

The points of $\mathbb{P}^1$ are represented by elements of $\mathbb{C} \setminus \{0\}$. We say the point represented by (x, y) is denoted $[x : y]$. For example, since $(1, 0) \sim (2, 0)$, we have $[1 : 0] = [2 : 0]$.

Exercise 2.5. Here is a general fact for quotient spaces. If $f : X \to Y$ is a continuous function and $\pi : X \to X/\sim$ is the quotient map. Then there exists an induced map $g : X/\sim \to Y$ such that $g \circ \pi = f$ if and only if $f(x) = f(y)$ whenever $x \sim y$. There are analogous statements in group theory that can inspire a proof of this fact.

As a general philosophy in algebraic geometry, one often want to study the functions on a space than the space itself. We also only care about polynomials in the sense that the functions on $\mathbb{P}^1$ we would like to study should be induced by polynomials on $\mathbb{C}^2 \setminus \{0\}$, i.e. functions of form $p : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ given by a polynomial $p(x, y)$. Such functions are called *regular functions* in the context of algebraic geometry.

However, in order for a polynomial $p(x, y)$ to induce a function on $\mathbb{P}^1$, we need $p(x, y) = p(kx, ky)$ for all $k \in \mathbb{C}$. The only such polynomials are constants, but these are not interesting. One compromise one may make is to allow *rational functions.*

You have seen in complex analysis that rational functions on $\mathbb{C}$ are of form f/g for polynomials f and $g \neq 0$, which are not necessarily defined on all of $\mathbb{C}$, and have poles at the zero set of g. In a similar manner, we can consider the functions x/y and y/x on $\mathbb{C}^2$, which induce functions on $\mathbb{P}^1$ with a pole at $[1 : 0]$ and $[0, 1]$, respectively. We call them the "slope functions" and say they are rational functions on $\mathbb{P}^1$.

Exercise 2.6. Look up the definition of rational functions on a variety in the context of algebraic geometry. Show that the function field of $\mathbb{P}^1$ is generated by x/y.

Now consider the map

$$\frac{x}{y} : \mathbb{P}^1 \setminus \{[1 : 0]\} \to \mathbb{C}.$$

This is a continuous bijection, and more generally it is a biholomorphism. You have seen that $\mathbb{C} \cup \{\infty\}$ can be viewed topologically as a sphere. If we extend x/y to all of $\mathbb{P}^1$ by sending $[1 : 0]$ to ∞, then we obtain a homeomorphism from $\mathbb{P}^1$ to a sphere. This is why we often call the Riemann surface $\mathbb{P}^1$ the *Riemann sphere.*

Exercise 2.7 (Another another definition)**.** As seen above, $\mathbb{P}^1$ minus $[1 : 0]$ is $\mathbb{C}$. Using the other slope function y/x, we see $\mathbb{P}^1$ minus $[0 : 1]$ is also $\mathbb{C}$. Conversely, if we take two copies of $\mathbb{C}$ and glue them together, we can obtain $\mathbb{P}^1$. Let $X_1 = \mathbb{C}, X_2 = \mathbb{C}$. Prove that

$$(X_1 \bigsqcup X_2)/\sim$$

is another valid definition of $\mathbb{P}^1$, where $z \sim w$ for any $z \in X_1 \setminus \{0\}, w \in X_2 \setminus \{0\}$ if $z = 1/w$.

2.3. **The torus $y^2 = x(x-1)(x-2)$.** We have seen previously the surface $C = \{(x, y) \in \mathbb{C}^2 | x^2 + y^2 = 1\}$ is topologically a tube (open on both ends). A tube, topologically, is the same as a sphere minus two points. Indeed, consider the map

$$\gamma : \mathbb{C} \cup \{\infty\} \setminus \{\pm i\} \to C$$
$$t \mapsto \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right), \quad \infty \mapsto (-1, 0).$$

This is a continuous bijection, and we know from previous section $\mathbb{C} \cup \{\infty\} \setminus \{\pm i\}$ is a sphere minus two points. They correspond the two "square roots" of $(1 - x^2)$ at $x = \infty$. If we add in these two points, then we may say a "compactification" of C is the sphere.

The picture we saw last time for the circle $x^2 + y^2 = 1$ is obtained by cutting open to copies of $\mathbb{C} \cong \mathbb{R}^2$ along an interval, then gluing them along the cut to obtain a tube. If we were to include the two points at infinity as in the last paragraph, then we would be cutting two spheres along an interval, then glueing them together into a new sphere.

Now consider the surface $E = \{(x, y) \in \mathbb{C}^2 | y^2 = x(x-1)(x-2)\}$. The map $E \to \mathbb{C}$ given by $(x, y) \mapsto y$ is a 2-covering with branched points at $x = 0, 1, 2$. Thus to see what E looks like, we glue together two copies of $\mathbb{C}$ along the branched cuts. One of the branch cut is the interval $[0, 1]$, and the other is $[2, \infty)$. It is easier for us to look at the compactification instead. Similar as before, we glue together two spheres, cutting open the intervals between $[0, 1]$, and $[2, \infty]$. In summary, we cut open two holes on each of the two spheres, which gives us two tubes, and glue along the corresponding sides, which gives us a torus.

Exercise 2.8. What does the surface $y^2 = (x-a)(x-b)(x-c)$ look like in general? What would go wrong if some of a, b, c coincide?

Exercise 2.9. Prove that for any polynomial $p(x) = ax^3 + bx^2 + cx + d$, there is a linear change of coordinates $y = Ax + B$ such that $p(y) = y^3 + Cx + D$. Further prove that p has three distinct roots if and only if $4C^3 + 27D^2 \neq 0$.

⟨defn:ellip⟩ **Definition 2.10.** An *elliptic curve* is the set of points in $\mathbb{C}^2$ such that $y^2 = x^3 + ax^2 + b$ for some fixed $a, b \in \mathbb{C}$ such that $4a^3 + 27b^2 \neq 0$.

2.4. **The Weierstrass function.** Now we shall make the connection between the tori $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $y^2 = x(x-1)(x+1)$. Consider the Weierstrass $\mathcal{P}$-function

$$\mathcal{P} : \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{C}$$
$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(z + m + in)^2} - \frac{1}{(m + in)^2}.$$

This is a meromorphic function. Note that it is technically defined on $\mathbb{C}$, but you can show it also induces a map on the quotient space $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

Exercise 2.11. As a function on $\mathbb{C}$, what are the poles of $\mathcal{P}$? As a function on $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, what are its poles?

Exercise 2.12. Consider the Laurent expansion of $\mathcal{P}$ at the origin $\sum c_i z^i$. What are the coefficients c_{-2}, c_{-1}, c_0? Conversely, suppose f is a meromorphic function on $\mathbb{C}$ that induces a map on $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, whose Laurent expansion at 0 has those three coefficients, then prove $f = \mathcal{P}$.

Exercise 2.13. Compute $\mathcal{P}'(z)$ and check the equality $\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - g_2\mathcal{P}(z) - g_3$ for some $g_2, g_3 \in \mathbb{C}$. You should get $g_2 = 4, g_3 = 0$.

With the above exercise, we may consider the map

$$\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{C}^2$$
$$z \mapsto (\mathcal{P}(z), \frac{1}{2}\mathcal{P}'(z)).$$

You may show this is an embedding, and the image of this map are points that satisfy $y^2 = x(x-1)(x+1)$, which is an elliptic curve. Note that this polynomial is not of the form in Definition 2.10. But it will be after a change of coordinates.

More generally, one can define the Weierstrass $\mathcal{P}$-function for a lattice $\Gamma \subseteq \mathbb{C}$, which will induce an elliptic curve in the same manner as above.

3. Week 3

3.1. **Compactification of the curve $x^2+y^2=1$.** In algebraic geometry, compact manifolds/varieties are often easier to deal with. We have seen the plane $\mathbb{C}$ can be compactified into the sphere $\mathbb{P}^1$ by adding the point at infinity.

In general, the projective space $\mathbb{P}^n_{\mathbb{R}}$ is compact, as it is the quotient space of the $(n+1)$-sphere in $\mathbb{R}^{n+1}$, which is compact, by the group $\{\pm 1\} = \mathbb{Z}_2$. One can also prove that $\mathbb{P}^n_{\mathbb{C}}$ is compact similarly.

In the spirit of $\mathbb{P}^1$ being topologically the same as $\mathbb{C}$ plus a point $\{\infty\}$, we can also see that $\mathbb{P}^1$ is given by $\mathbb{C}^2$ plus $\mathbb{C}$ plus $\{\infty\}$ as follows.

The points of $\mathbb{P}^2$ are lines in $\mathbb{C}^3$, which are equivalence classes $[X : Y : Z]$ under the relation $(X, Y, Z) \sim k(X, Y, Z)$ for any $k \in \mathbb{C}$. Consider the set $\{Z = 0\} \subseteq \mathbb{P}^2$. This is the set of points $[X : Y : 0]$, which is biholomorphic to the space $\mathbb{P}^1$ via the map $[X : Y : 0] \mapsto [X : Y]$. On the other hand, the set $\{Z \neq 0\} \subseteq \mathbb{P}^2$ can be identified with $\mathbb{C}^2$ by the map $[X : Y : Z] \mapsto (\frac{X}{Z}, \frac{Y}{Z})$. Hence $\mathbb{P}^2$ is $\mathbb{C}^2$ plus $\mathbb{P}^1$, which is $\mathbb{C}^2$ plus $\mathbb{C}^1$ plus $\{\infty\}$.

Exercise 3.1. More generally, prove that $\mathbb{P}^n$ can be decomposed into $\mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{\infty\}$.

Since $\mathbb{P}^2$ contains a copy of $\mathbb{C}^2$ in it, we can try to extend the curve $C = \{x^2 + y^2 = 1\} \subseteq \mathbb{C}^2$ to a curve $\overline{C} \subseteq \mathbb{P}^2$. Via the correspondence above, the function $x^2 + y^2 = 1$ in $\mathbb{C}^2$ corresponds to $(\frac{X}{Z})^2 + (\frac{Y}{Z})^2 = 1$ in the subset $\{Z \neq 0\} \subseteq \mathbb{P}^2$. This is equivalent to the function $X^2 + Y^2 = Z^2$, which is defined in all of $\mathbb{P}^2$. Therefore the curve

$$\overline{C} := \{X^2 + Y^2 = Z^2\} \subseteq \mathbb{P}^2$$

is the closure of the image of C by the map $C \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$. In particular, $\overline{C}$ is compact. This is why we call $\overline{C}$ the *compactification of* C in $\mathbb{P}^2$.

We have seen before C is topologically a tube. The extra points in $\mathbb{P}^2$ we added to C to obtain $\overline{C}$ are the points in $\{Z = 0\} \cong \mathbb{P}^1$ such that $X^2 + Y^2 = 0$. These are the two points $[1 : \pm i]$, which we regard as two points at "infinity". Adding two infinity points to the two ends of the tube, we see $\overline{C}$ is topologically sphere, which again is indeed compact.

Exercise 3.2. Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, which cuts out a hypersurface S in $\mathbb{C}^n$, what is the polynomial that defines the function in $\mathbb{P}^{n+1}$ which cuts out its compactification $\overline{S}$? The process of getting this polynomial is called *homogenization*.

3.2. **Compactification of the curve $y^2 = x(x-1)(x+1)$.** Previously we have called the complex curve $E := \{y^2 = x(x-1)(x+1)\}$ a torus (call it E for elliptic). But the curve itself is not compact, while a torus is. Now with the notion of compactification, we can make sense of this.

The normalization of E is given by the polynomial $Y^2 Z = X(X - Z)(X + Z)$ in $\mathbb{P}^2$. On the locus $\{Z \neq 0\}$, this polynomial cuts out the original curve E as we have seen before, and the extra points added to compactify it is in $\{Z = 0\}$, satisfying $0 = X^3$. Thus the compactification $\overline{E}$ of E is given by E plus the point $[0 : 1 : 0]$.

To get a picture of E, we first pick the branch cut given by removing the interval $[-1, 0]$ and the interval $[1, \infty)$. Regarding $[0 : 1 : 0]$ as the point at infinity, we add in two infinities to the two branches. After glueing, the two infinities will be identified as one point, for the same reason the points $-1, 0, 1$ in the two copies of the branches identify as a single point after gluing.

After adding the point at infinity to the two branches, we obtain two copies of a sphere with two intervals cutted out. Glueing along the cuts, we indeed obtain a torus.

Exercise 3.3. In class, you saw a picture of $y^2 = x(x - 1)(x - 2)(x - 3)$. Now come up with pictures for the compactification of $y^2 = x(x - 1) \cdots (x - n)$ in $\mathbb{P}^2$. This should split into cases where n is odd or even.

3.3. **Lines of $\mathbb{P}^2$ are planes of $\mathbb{C}^3$ through the origin.** In class you saw the statement "lines of $\mathbb{P}^2$ correspond to planes of $\mathbb{C}^3$ through the origin". The following exercises help you figure out how to interpret this formally. Instead of over $\mathbb{C}$, we will work over $\mathbb{R}$ for better visualization, then formally we can just switch all the $\mathbb{R}$ to $\mathbb{C}$ in our argument and make sure every step still works algebraically.

Exercise 3.4. First the space $\mathbb{P}^2_{\mathbb{R}}$ is a non-orientable surface, so we need to make sense of "lines in $\mathbb{P}^2$". Give a definition for what a line is in $\mathbb{P}^2$.

Exercise 3.5. Let V be a plane in $\mathbb{R}^3$ that pass through the origin, i.e. a linear subspace of dimension 2. Show that the image of V via the map $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2$ is a line by your previous definition.

Exercise 3.6. Finally given a line L in $\mathbb{P}^2$, show that the preimage of L by π is a linear subspace of dimension 2 in $\mathbb{R}^3$.