PMATH965 WRITTEN PROJECT VIRTUAL FUNDAMENTAL CLASSES

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Contents

1. Introduction

In this note we review the construction of virtual fundamental classes via perfect obstruction theories by Behrend-Fantechi [BF98]. We will see some examples of obstruction theories from deformation theory, and methods of dealing with virtual classes such as virtual pullback and localization. In the end we discuss the virtual classes on Calabi-Yau 4-folds by Borisov-Joyce [BJ15] and Oh-Thomas [OT23] using shifted symplectic structures.

2. Behrend-Fantechi Virtual fundamental classes

In enumerative geometry, counts of objects of certain properties are usually obtained by integrating the cohomology classes corresponding to those properties in the moduli space of such objects. As moduli schemes are usually not smooth or even reduced, there is no deformation invariant fundamental class to integrate against. By the notion of virtual smoothness, this problem can be resolved by associating the moduli space a homology class of expected dimension, called the virtual fundamental class, introduced by Li-Tian [LT96] and Behrend–Fantechi [BF98]. In this section we follow the construction given in [BF98] and [BCM18].

2.1. **Intrinsic normal cone.** Let X be a scheme (or Deligne-Mumford stack). A *cone* over X is a scheme $C = \operatorname{Spec}_X S^\bullet$, where $S^\bullet$ is a sheaf of graded $\mathcal{O}_X$-algebras such that $\mathcal{O}_X = S^0$ and S^1 is coherent and generates $S^\bullet$ as an algebra. If $k = \mathbb{C}$ is the base field, the action $t \cdot S^i$ by multiplication of t^i induces an $\mathbb{A}^1$-action on C. The *zero section* of C is $0 : X \to C$ given by the surjection $S^\bullet \twoheadrightarrow S^0$ which realizes X as a subscheme of C.

For a locally free sheaf $\mathcal{E}$, the cone $C(\mathcal{E}^\vee) = \operatorname{Spec}\operatorname{Sym}\mathcal{E}^\vee$ is a vector bundle over X. In general, a cone of form $\operatorname{Spec}\operatorname{Sym}\mathcal{F}$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a group scheme and called an *abelian cone*. For any cone $C = \operatorname{Spec} S^\bullet$, the *abelianization* of C is given by $A(C) = \operatorname{Spec}\operatorname{Sym} S^1$.

Natural examples of cones are tangent cones of points, or more generally the *normal cone* of a closed embedding $X \hookrightarrow Y$. This is given by

$$C_{X/Y} = \operatorname{Spec}_X \bigoplus_{n\geq 0} (I^n/I^{n+1})$$

where I is the ideal sheaf of X in Y. In the case of a regular embedding, the normal cone coincides with its abelianization, the normal sheaf (bundle).

Let $X \hookrightarrow Y$ be a closed embedding. Suppose $E_{X/Y}$ is a vector bundle of rank r and $i : C_{X/Y} \to E_{X/Y}$ is an embedding of cones (preserves 0 section and the $\mathbb{A}^1$-action). Then $E_{X/Y}$ is called an obstruction theory and the virtual fundamental class with respect to $E_{X/Y}$ is defined by

$$[X]^{\mathrm{vir}} = 0^!_{E_{X/Y}} i_*[C_{X/Y}]$$

where i_* is the proper pushforward and $0^!_{E_{X/Y}} : A_*(E_{X/Y}) \to A_{*-r}(X)$ is given by the inverse map of the flat pull-back [Ful13, Theorem 3.3]. In other words, the virtual fundamental class with respect to the obstruction theory $E_{X/Y}$ is given by intersecting the normal cone $C_{X/Y}$ with the 0-section of $E_{X/Y}$. However, as the dimension of $[X]^{\mathrm{vir}}$ depends on the rank of $E_{X/Y}$, this construction will usually not give us a class of the expected dimension. Thus we consider a more general setting to impose more restrictions to the obstruction theory.

Let X be a Deligne-Mumford stack. Suppose E is a vector bundle, then a cone morphism $E \to C$ is called an *E-cone* if the action of E on $A(C)$ preserves C. This induces an action of E on C, and the stack quotient $[C/E]$ naturally admits a 0-section and an $\mathbb{A}^1$-action. This is a cone stack in the sense that the morphism $C \to [C/E]$ is $\mathbb{A}^1$-equivariant, smooth and surjective. The natural example we have is the intrinsic normal cone $\mathfrak{C}_X$ as a subcone stack of the intrinsic normal sheaf $\mathfrak{N}_X$. These can be described via local embeddings of X. Suppose $U \to X$ is affine étale, and $U \to V$ an embedding with V smooth. We have

$$\mathfrak{C}_X = [C_{U/V}/T_V|_U], \quad \mathfrak{N}_X = [N_{U/V}/T_V|_U].$$

Let $E^\bullet \in D(\mathrm{QCoh}(X))$ be a complex of quasi-coherent sheaves on X concentrated in the non-positive degrees. Define $h^1/h^0(E^\bullet)$ as the stack theoretic quotient

$$[\ker(E^1 \to E^2)/\operatorname{coker}(E^0 \to E^1)]$$

Let $L_X^\bullet \in D^{\leq 0}(\mathrm{QCoh}(X))$ be the cotangent complex of X, then $h^1/h^0((L^\bullet)^\vee)$ is an algebraic cone stack over X called the *intrinsic normal sheaf* $\mathfrak{N}_X$.

Definition 2.1. An *obstruction theory* for X is a complex $E \in D^{\leq 0}(\mathrm{QCoh}(X))$ such that E^0 and E^{-1} are coherent, together with a morphism $\phi : E^\bullet \to L_X^\bullet$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. When $E^\bullet = [E^{-1} \to E^0]$ is a 2-term complex of vector bundles on X, we say $E^\bullet$ is a *perfect obstruction theory.*

The morphism ϕ of an perfect obstruction theory induces a closed embedding

$$\phi^\vee : \mathfrak{N}_X \to \mathfrak{E} := h^1/h^0((E^\bullet)^\vee) = [E_1/E_0].$$

where $T^{\mathrm{vir}} = E_\bullet = (E^\bullet)^\vee$ is the *virtual tangent bundle*. The term E_0 is to be understood as the "tangent" and the term E_1 is to be understood as the "obstruction". Recall that we would like the virtual class to be the intersection of the normal cone with the zero section of the obstruction.

Definition 2.2. The *virtual fundamental class* is

$$[X]^{\mathrm{vir}} = [X, E^\bullet] = 0^!_{E_1}[\mathfrak{C}_X \times_{\mathfrak{E}} E_1].$$

The dimension of $[X^{\mathrm{vir}}]$ is equal to $\mathrm{rank}(E^\bullet) = \mathrm{rank}\, E^0 - \mathrm{rank}\, E^{-1}$, which is called the *virtual dimension* of X with respect to $E^\bullet$.

2.2. **Examples of obstruction theory.** In this section we go through some examples of obstruction theories of the moduli spaces.

2.2.1. *Zero locus of a section.* The easiest case of an obstruction bundle is when X is the zero locus of a section of some vector bundle $E \to Y$. If E has rank r, the virtual fundamental class of X is of dimension $n - r$. If the ideal of X in Y is I, then the natural surjection $E^\vee|_X \to I/I^2$ gives an obstruction theory

$$[E^\vee|_X \xrightarrow{0} \Omega_Y|_X] \to L^\bullet_X = [\ldots \to I/I^2 \to \Omega_Y|_X].$$

In this case the pushforward of $[X]^{\mathrm{vir}}$ to Y is given by $e(E) \cdot [Y]$ where e is the Euler class.

2.2.2. *Moduli space of stable bundles on curves.* Let $M^s(r, d)$ be the moduli space of stable bundles of rank r and degree d of a smooth projective variety X. One can check whether a complex gives obstruction theory using deformation theory according to [BF98, Theorem 4.5]. In this example we shall see an obstruction theory is given by $R\pi_* R\mathscr{H}om(\mathcal{F}, \mathcal{F})$ by computing deformations of vector bundles [HL10, Section 2.A.6], where $\pi : M^s(r, d) \times X \to M^s(r, d)$ is the projection and $\mathcal{F}$ is the universal bundle. Recall that the universal sheaf is only unique up to some twist in the Picard group, but this twist will cancel in $R\mathscr{H}om(\mathcal{F}, \mathcal{F})$.

Definition 2.3. Let $\mathcal{A}rt$ be the category of Artin local rings. Let $D : \mathcal{A}rt \to \mathcal{S}et$ be a functor, and T, O be finite dimensional vector spaces. We say D admits a *deformation-obstruction theory* with respect to the tangent T and obstruction O if the following holds. Let $A' \twoheadrightarrow A$ be a small extension, that is it has nilpotent kernel I. Then we have

(1) a $T \otimes_{\mathbb{C}} I$ action on $D(A')$,
(2) an obstruction map $\mathfrak{o} : D(A) \to O \otimes_{\mathbb{C}} I$.

such that the image of $D(A') \to D(A)$ is equal to $\mathfrak{o}^{-1}(0)$, and every non-empty fiber of $D(A') \to D(A)$ is a torsor under the action of $T \otimes_{\mathbb{C}} I$. The data is also required to be functorial among different small extensions in the sense of [Fan05, Definition 6.1.21].

The tangent-obstruction theory of $M^s(r, d)$ at a fixed stable bundle F is given by tangent space $\mathrm{Ext}^1(F, F)$ and obstruction space $\mathrm{Ext}^2(F, F)$. When X is a curve, $\mathrm{Ext}^2(F, F)$ vanishes for dimension reason, and the moduli space is smooth. When X is a surface, we obtain a perfect obstruction theory. We give a sketch of the proof below.

Define D_F the deformation functor at F by

$$D_F(A) := \{(\mathcal{F}, \phi) | \mathcal{F} \text{ is an } A\text{-flat family of bundles on } X \text{ and } \phi : \mathcal{F} \otimes_A (A/m) \cong F\}$$

where m is the maximal ideal of the Artin ring A. Let $A' \twoheadrightarrow A$ be a small extension. Let $(\mathcal{F}, \phi) \in D_F(A)$. The fiber of $D_F(A') \to D_F(A)$ are the deformations over A' that restrict to $\mathcal{F}$. Let U_i be a open cover of X such that the restrictions $\mathcal{F}_i$ is free over $\mathrm{Spec}(A) \times U_i$, so the extension to $\mathrm{Spec}(A') \times X$ are necessarily free over each $\mathrm{Spec}(A') \times U_i$. Let $\mathcal{F}_i'$ be the corresponding free sheaves, and $g_{ij} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ the gluing maps. To obtain a lift $\mathcal{F}'$, we need gluing maps

$$g_{ij}' : \mathcal{F}_i'|_{U_{ij}} \to \mathcal{F}_j'|_{U_{ij}}$$

that and lift each g_{ij} and satisfy the cocycle condition. Choose any such lifting, and set

$$\delta_{ijk} = (g_{ik}')^{-1} g_{jk}' g_{ij}'.$$

Since the maps g_{ij} satisfy the cocycle condition, we know δ lifts the identity on $\mathcal{F}_i|_{U_{ijk}}$. The "obstruction" is described by how far away δ_{ijk} is from the identity on $\mathcal{F}_i'|_{U_{ijk}}$. Tensor the exact sequence $0 \to I \to A' \to A \to 0$ by $\mathcal{F}_i'|_{U_{ijk}}$, we get

$$0 \to I \otimes F_i|_{U_{ijk}} \to \mathcal{F}_i'|_{U_{ijk}} \to \mathcal{F}_i|_{U_{ijk}} \to 0$$

where first term on the left is obtained by $I \otimes A/m = I/mI = I$. Therefore the map $\delta_{ijk} - \mathrm{Id}$ sends $\mathcal{F}_i'|_{U_{ijk}}$ to $I \otimes F_i|_{U_{ijk}}$ and we have

$$\delta_{ijk} - \mathrm{Id} \in \mathrm{Hom}(\mathcal{F}_i'|_{U_{ijk}}, I \otimes F_i|_{U_{ijK}}) = \mathrm{Hom}(F_i|_{U_{ijK}}, F_i|_{U_{ijK}}) \otimes I,$$

and the equality is again obtained from $I \cdot m = 0$. Moreover, these maps satisfy the 2-cocycle condition and give a class

$$\mathfrak{o}(\mathcal{F}, \phi) := \{\delta - \mathrm{Id}\} \in \mathrm{Ext}^2(F, F) \otimes I.$$

In particular, one can check that this class does not depend on the lifting g_{ij}' or the covering $\{U_i\}$. Thus we obtain an obstruction map

$$\mathfrak{o} : D_F(A) \to \mathrm{Ext}^2(F, F) \otimes I.$$

If $\mathfrak{o}(\mathcal{F}, \phi) = 0$, then we can modify the liftings g_{ij}' so that they satisfy the cocycle condition, thereby giving a lifting $\mathcal{F}'$ on $\mathrm{Spec}(A') \times X$. Hence $\mathrm{Ext}^2(F, F)$ is the obstruction space.

For the tangent space, we need to consider the fibers of $D(A') \to D(A)$. Suppose $(\mathcal{F}', \phi'), (\mathcal{F}'', \phi'') \in D(A')$ are two liftings of $(\mathcal{F}, \phi)$. Choose local isomorphisms $g_i : \mathcal{F}_i' \mathcal{F}_i''$ and set $\delta_{ij} = g_j^{-1} \circ g_i$. By a similar argument as before we obtain a class $\{\delta_{ij} - \mathrm{Id}\} \in \mathrm{Ext}^1(F, F) \otimes I$, and the vanishing of this class corresponds to isomorphism between $\mathcal{F}'$ and $\mathcal{F}''$. Hence the elements in the fiber $D(A') \to D(A)$ are in bijection with $\mathrm{Ext}^1(F, F) \otimes I$, which gives us a torsor structure and the tangent space is $\mathrm{Ext}^1(F, F)$.

2.3. **Virtual pullback.** An example of dealing with virtual classes is via the virtual pullback. We recall the construction and application introduced in [Man08]. Let X be a Deligne-Mumford stack and $\mathfrak{E}$ a vector bundle stack of virtual rank d. Suppose $f : X \to Y$ is a morphism of Deligne-Mumford stacks (more generally, a Deligne-Mumford type morphism of algebraic stacks) and the relative intrinsic normal cone $i : \mathfrak{C}_{X/Y} \hookrightarrow \mathfrak{E}$ is a closed embedding. The *virtual pullback* $f^!_{\mathfrak{E}}$ is given by

$$A_*(Y) \xrightarrow{\sigma} A_*(\mathfrak{C}_{X/Y}) \xrightarrow{i_*} A_*(\mathfrak{E}) \xrightarrow{0^!_{\mathfrak{E}}} A_{*-d}(X)$$

where the first map is given by $\sigma([V_i]) = [\mathfrak{C}_{V \times_Y X/V}]$, and the last map is the canonical isomorphism for vector bundle stacks.

In the case of a fiber diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

the virtual pullback $f^!_{\mathfrak{E}}$ is

$$A_*(Y') \xrightarrow{\sigma} A_*(\mathfrak{C}_{X'/Y'}) \cong A_*(p^*\mathfrak{C}_{X/Y}) \xrightarrow{i_*} A_*(p^*\mathfrak{E}) \xrightarrow{0^!_{p^*\mathfrak{E}}} A_{*-d}(X').$$

This is a generalized version of Gysin morphism, and in the case where $\mathcal{E}$ arise from a perfect obstruction theory, it satisfies similar properties as the Gysin morphism described below.

Theorem 2.4 ([Man08]Theorem 4.1)**.** *In the setting above, suppose we have fiber diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y \end{array}$$

the virtual pullback satisfies the following property.

(1) If p is proper and $\alpha \in A_k(Y'')$, then

$$f^!_{\mathfrak{E}} p_*(\alpha) = q_* f^!_{\mathfrak{E}}(\alpha) \in A_{k-d}(X').$$

(2) If p is flat or of relative dimension n, $\alpha \in A_k(Y')$, then

$$f^!_{\mathfrak{E}} p^*(\alpha) = q^* f^!_{\mathfrak{E}}(\alpha) \in A_{k+n-d}(X'').$$

(3) If $\alpha \in A_k(Y'')$, then

$$f^!_{\mathfrak{E}} \alpha = (f')^!_{g^*\mathfrak{E}} \alpha \in A_{k-d}(X'').$$

The virtual pullback allows the computation of virtual classes on a complicated space through the virtual class on a simpler space. Suppose $f : X \to Y$ and a distinguished triangle of (relative) perfect obstruction theories

$$f^*E^\bullet_Y \to E^\bullet_X \to E^\bullet_{X/Y} \to f^*E^\bullet_Y[1]$$

compatible with the natural distinguished triangle of cotangent complex

$$f^*L^\bullet_Y \to L^\bullet_X \to L^\bullet_{X/Y} \to f^*L^\bullet_Y[1].$$

Then by [Man08, Corollary 4.9], the virtual class for F is

$$[F]^{\mathrm{vir}} = f^!_{\mathfrak{E}_{F/G}}[G]^{\mathrm{vir}}.$$

Example 2.5. In application Manolache first showed that for $\mathbb{P}$ a homogeneous space and $\tilde{\mathbb{P}}$ a blow up, the virtual class of the moduli space $\overline{M}_{0,n}(\tilde{\mathbb{P}}, \tilde{\beta})$ of stable maps with n marked points, of genus 0 and homology class $\tilde{\beta}$, pushes to the virtual class of $\overline{M}_{0,n}(\mathbb{P}, \beta)$ where $\tilde{\beta}$ lifts β. Using virtual pullback, it then follows that this holds for any smooth projective subvariety X of $\mathbb{P}$. More precisely, if Z is another subvariety intersecting X transversely and $p : \tilde{X} \to X$ is the blow up of X along $X \cap Z$, then

$$\overline{p}_*[\overline{M}_{0,n}(\tilde{X}, \tilde{\beta})]^{\mathrm{vir}} = [\overline{M}_{0,n}(X, \beta)]^{\mathrm{vir}}.$$

As a consequence, one obtain the equality between the Gromov-Witten invariants of cohomology classes γ_i on X and that of $p^*\gamma_i$ on the blow up $\tilde{X}$. ◇

2.4. **Virtual localization.** Other than virtual pullbacks, localization is also often used in computation of virtual invariants. This is the virtual version of the equivariant localization [EG95] which states that if X is a toric variety, for a cycle α in the equivariant Chow ring $A_*^T(X)$, we have

$$\int_X \alpha = \sum_F \int_F \frac{\alpha|_F}{e^T(N_F)}$$

where the sum goes through the components of the fixed locus of X, π is the pushforward to a point, and e^T is the equivariant Euler class. This is well defined as $e^T(N_F X)$ can be shown to be invertible in the localized ring $A_*^T(X, \mathbb{Q})[t^{-1}]$, where t_i are the generators of the equivariant ring of T.

Suppose X is a toric variety with a perfect obstruction theory $(E^\bullet, \phi)$ such that $E^\bullet$ and ϕ are equivariant. In the construction of the virtual fundamental class, the cones used are T-equivariant, which give us an equivariant virtual fundamental class in the equivariant Chow group $A_d^T(X)$. The T-fixed part of $E^\bullet$ is a perfect obstruction theory when restricted to each component of the fixed locus, and induces virtual classes. The virtual localization formula is

$$\int_{[X]^{\mathrm{vir}}} \alpha = \sum_F \int_{[F]^{\mathrm{vir}}} \frac{\alpha|_F}{e^T(N_F^{\mathrm{vir}})}$$

where the virtual normal bundle $N^{\mathrm{vir}}F$ is given by the T-moving part of the virtual tangent bundle $T^{\mathrm{vir}}|_F$.

In the computation of Gromov-Witten invariants, the fixed loci for the moduli space of stable maps are described by graphs, the virtual class on each loci coincides with the usual fundamental class, and the Euler class of N^{vir} is expressed by tautological classes, making the localized integrals very computable. In the case of Donaldson-Thomas invariants, the fixed loci of Quot schemes are locally described by coloured partitions, allowing the use of many combinatorial methods.

3. Oh-Thomas Virtual fundamental classes

3.1. **Virtual fundamental class.** In the study of moduli of sheaves on Calabi-Yau 4-folds, the obstruction theory are often not perfect, but has 3 terms. J.Oh and R.P.Thomas constructed virtual fundamental classes using local models of derived stacks from shifted symplectic structures. We follow the construction given in [OT23, Section 4].

Let $(X, \mathcal{O}_X(1))$ be a smooth projective Calabi-Yau 4-fold and $M = M^s(c)$ the projective moduli space of $\mathcal{O}_X(1)$-Gieseker stable sheaves with Chern character c, where c is a class such that all semi-stable sheaves are stable. An obstruction for M is given by [HT14, Theorem 4.1]

$$E^\bullet = R_{\pi*} R\mathscr{H}om(\mathcal{F}, \mathcal{F})_0[3] \xrightarrow{At} L_M^\bullet$$

where $\pi : M \times X \to M$ is the projection, $\mathcal{F}$ is the universal sheaf and $(\cdot)_0$ denotes the trace-free part. By Serre duality, the complex $E^\bullet$ is self dual in the sense that we have an isomorphism

$$\theta : E^\bullet \to E^\vee[2].$$

This induces an isomorphism

$$Q : \det(E^\bullet) \otimes \det(E^\bullet) \cong \mathcal{O}_M$$

and a choice of square root of this isomorphism is called an orientation. Moreover, $E^\bullet$ is quasi-isomorphic to a self dual complex

$$[T \to E \to T^*]$$

concentrated in degree $-2, -1, 0$. For a fixed orientation, the middle term E can be regarded as a $SO(r, \mathbb{C})$ bundle. Truncating the complex to $[E \to T^*]$ will result in a perfect obstruction theory (but of wrong virtual dimension), so from the usual construction of virtual classes, we obtain a cone

$$C_{E^\bullet} = \mathfrak{C}_M \times_{\mathfrak{E}} E^* \subseteq E^* \cong E.$$

Using explicit local models of derived stacks with shifted symplectic structure given by [BBBJ13] (which are versions of Darboux charts for derived stacks), $C_{E^\bullet}$ is shown to be isotropic in the sense that the quadratic form

$$q : E \otimes E \to \mathcal{O}_Y$$

vanishes on $C_{E^\bullet}$. For such a cone, a square root Gysin map $\sqrt{0^!_E}$ is defined using the square root Edidin-Graham class of $SO(r, \mathbb{C})$-bundles, where the $\mathbb{Z}[1/2]$ coefficients are due to taking "square roots" of a bundle, hence its name. From here, a virtual fundamental class of dimension $\frac{1}{2}\mathrm{vdim}(E^\bullet)$ is obtained as

$$[M]^{\mathrm{vir}} = \sqrt{0^!_E}[C_{E^\bullet}] \in A_{\frac{1}{2}\mathrm{vdim}}(M, \mathbb{Z}[1/2]).$$

As the construction only relies on the (-2)-shifted symplectic structure and an orientation, the virtual structure exists in more generality.

3.2. **Basic properties.** As the cone $C_{E^\bullet}$ is obtained from the truncated complex $[E \to T^*]$, the virtual structure sheaf is not well defined unless the omitted term is accounted for. Let $\mathcal{O}^{\mathrm{vir}}$ be the virtual structure sheaf obtained from the perfect obstruction theory $[E \to T^*]$, then the *twisted structure sheaf* for the obstruction $E^\bullet$ is given by [OT23, (105)]

$$\hat{\mathcal{O}}^{\mathrm{vir}} = (-1)^{\frac{1}{2}\mathrm{vdim}} \mathcal{O}^{\mathrm{vir}} \cdot \sqrt{K^{\mathrm{vir}}} \in K_0(M, \mathbb{Z}[1/2]).$$

Here K^{vir} is given by $\det[\Lambda \to T^*]$ where Λ is a maximal isotropic subbundle of E (thought of as a "square root" of E), and the square root is taken in the Grothendieck group $K_0(M, \mathbb{Z}[1/2])$ with half coefficients.

The Oh-Thomas virtual fundamental class and virtual structure sheaf satisfy similar properties as the Behrend-Fantechi virtual structures. We shall go over two examples.

3.2.1. *Virtual Riemann-Roch.* For a quasi-projective scheme M, [Ful13, Corollary 18.3.2] provides an isomorphism between K-theory and the Chow homology by

$$\tau_M : K_0(M)_{\mathbb{Q}} \to A_*(M)_{\mathbb{Q}}.$$

The usual virtual Riemann-Roch theorem is given by [FG07], which states

$$\tau_M(\mathcal{O}^{\mathrm{vir}}) = \mathrm{td}(T^{\mathrm{vir}}) \cap [M]^{\mathrm{vir}}.$$

The Oh-Thomas virtual Riemann-Roch theorem [OT23, Theorem 6.1] states

$$\tau_M(\hat{\mathcal{O}}^{\mathrm{vir}}) = \sqrt{\mathrm{td}}(T^{\mathrm{vir}}) \cap [M]^{\mathrm{vir}}.$$

where $\sqrt{\mathrm{td}}(T^{\mathrm{vir}}) = \mathrm{td}((K^{\mathrm{vir}})^\vee)\,\mathrm{ch}\sqrt{\det K^{\mathrm{vir}}}$. Setting *twisted virtual Euler characteristic*

$$\chi^{\mathrm{vir}}(-) := \chi(M, - \otimes \hat{\mathcal{O}}^{\mathrm{vir}}),$$

we obtain a virtual version of Hirzebruch-Riemann-Roch Theorem: for $V \in K_0(M)$,

$$\chi^{\mathrm{vir}}(V) = \int_{[M]^{\mathrm{vir}}} \sqrt{\mathrm{td}}(T^{\mathrm{vir}})\,\mathrm{ch}(V).$$

3.2.2. *Virtual localization.* When the Calabi-Yau 4-fold X admits a torus action by Y, the T-action lifts to the universal sheaf, which makes the obstruction theory T-equivariant, and a virtual localization formula is given by

$$[M]^{\mathrm{vir}} = \sum_F \iota_* \frac{[F]^{\mathrm{vir}}}{\sqrt{e_T}(N^{\mathrm{vir}})} \in A^T_{\frac{1}{2}\mathrm{vdim}}(M, \mathbb{Q})[t^{-1}]$$

where similar to the Graber-Pandharipande localization, the sum goes through fixed components, ι denotes the inclusion in M, N^{vir} is the moving part of T^{vir}, and $\sqrt{e_T}$ is the equivariant version of the square root Edidin-Graham Euler class.

The construction of the virtual structure sheaf allowed K-theoretic invariants for Calabi-Yau 4-folds to be formalized, as they were only predicted to exist in for instance [CKM22]. An example for virtual localization is the computation of Donaldson-Thomas invariants for Calabi-Yau 4-folds, where a vertex formalism performed in [CK20, Section 2.4], similar to that of [MNOP06] for Calabi-Yau 3-fold.

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