Hyperbolic group talk by Jiahui Huang

A. δ-hyperbolicity

The notion of hyperbolicity is usually associated with curvature in the study of manifolds. We would like to generalize this condition to metric spaces in geometric group theory, to apply properties to, for example, the Cayley graph of a group.

One observes that a hyperbolic triangle has sides very close to each other.[insert picture] This illustrates the property that two geodescis starting at the same vertex diverge quickly: the shortest path between two vertices is one that has to backtrack to somewhere near the third vertex.

Definition 1.1. Let X be a geodesic metric space. A geodesic triangle $[x, y, z]$ is δ-*thin* if for each point p on the edge $[x, y]$, there is a point q on edges $[x, z], [y, z]$ such that $d(p, q) \leq \delta$, and the property holds similarly for the other edges. X is δ-*hyperbolic* if every triangle is δ-thin. A group is *hyperbolic* if its Cayley graph is hyperbolic.

The specific value of δ is not important since the point is about the existence of some δ. We will see next week that quasi-isometries preserves hyperbolicity.

Example 1.2. The triangles in trees are in shape of three legs attached to a single vertex. Each side of a triangle is contained in the union of the other two sides. Therefore trees are 0-hyperbolic. In particular, the Cayley graph of a free group is hyperbolic.

Example 1.3. Since any finite group has bounded Cayley graph, they are all hyperbolic.

Example 1.4. Some metric spaces, for example the p-adic numbers, satisfy the property that the longest two sides of any triangle have equal length. Spaces with such property are all 0-hyperbolic. This can be shown using *Gromov product*.

Example 1.5. $\mathbb{R}^2$ is not hyperbolic. Therefore $\mathbb{Z}^2$ is not hyperbolic since its Cayley graph is quasi-isometric to $\mathbb{R}^2$. It is well known that if a group contains $\mathbb{Z}^2$ as a subgroup, then it is not hyperbolic.

Example 1.6. Draw the Cayley graph of the projective special linear group

$$PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I, -I\}$$

where

$$SL(2, \mathbb{Z}) = \{M \in M_2\mathbb{Z} : \det(M) = 1\}$$

Observe the Cayley graph is quasi-isometric to a tree, implying $PSL(2, \mathbb{Z})$ is hyperbolic. Since it is the quotient of $SL(2, \mathbb{Z})$ by a finite subgroup, the quotient map induces a quasi-isometry between their Cayley graphs. Therefore $SL(2, \mathbb{Z})$ is also hyperbolic.

Example 1.7. Draw fundamental group of T^2 and see that it is hyperbolic.

Example 1.8. Recall from previous talks that if a group acts geometrically on a space X, then it is quasi-isometric to X. Therefore if a group acts geometrically on any hyperbolic space, it is itself hyperbolic.

We know $SL(2, \mathbb{Z})$ acts on the upper half plane model of the hyperbolic space by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}$$

Can we conclude from this that $SL(2, \mathbb{Z})$ is hyperbolic? [insert picture here]

B. Word problem for hyperbolic groups

Recall the word problem is solvable if there is an algorithm to determine whether a word represents the identity. One can easily construct such an algorithm for groups such as $F_2 = \langle a, b\rangle$ or $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1}\rangle$ because of their simple presentation. We shall define a specific way to present a group, give an algorithm to solve the word problem for groups of that form, and show hyperbolic groups have such presentation.

Definition 1.9. Let S be a generating set of group G. A word in S is *reduced* if it does not contain any occurrences of aa^{-1} for $a \in S \cup S^{-1}$. A finite presentation $G = \langle S|R\rangle$ is a *Dehn presentation* if it is of the following form

- $R = \{r_1, ..., r_m\}$ consists of words of form $r_i = u_i v_i^{-1}$ (v_i may be empty).
- For each i, the word v_i is (strictly) shorter than u_i.
- If w is a reduced word in S that represents the identity,it is either empty or contains some u_i or u_i^{-1} as a sub-word.

Example 1.10. $F_2 = \langle a, b \rangle$ is a Dehn presentation because any word that represents the identity can be reduced to the empty word by removing occurrences of $aa^{-1}, bb^{-1}, a^{-1}a, b^{-1}b$.

Example 1.11. $PSL(2,\mathbb{Z}) = \langle a, b | \alpha^2, b^3 \rangle$ is a Dehn presentation. We take $u_1 = a^2, v_1 = 1, u_2 = b^2, v_2 = b^{-1}$. The third condition is satisfied because we see from the Cayley graph that a reduced word that represents the identtity, removing all the occourances of $u_1 = a^2$, gives loop which must contain a loop around one of the triangles. This implies a subword of form $u_2 = b^2$.

If a group has a Dehn presentation, then the word problem is solvable in linear time (with respect to word length). The algorithm is as follows: reduce w, find a sub-word u_i, replace u_i by v_i, and repeat. Since the word length is finite, we either get the empty word in at most that many steps, or a sub-word of form u_i can not be found, in which case w must not represent identity.

Definition 1.12. Let X be a metric space. A *geodesic* segment is an isometric embedding $[a, b] \to X$ where the interval $[a, b]$ has the usual metric of the real line. A *c-local geodesic* is a path $\gamma : [a, b] \to X$ such that every sub-path of length c is geodesic. This means for every $t, t' \in [a, b]$ with $|t - t'| \leq c$,

$$d(\gamma(t), \gamma(t')) = |t - t'|$$

Lemma 1.13. *Let X be a δ-hyperbolic space. If $\gamma : [0, L] \to X$ is a 8δ-local geodesic, and $\gamma' : [0, L'] \to X$ is a geodesic segment from $\gamma(0)$ to $\gamma(L)$, then for every point p on γ, there is some q on γ' with $d(p, q) < 2\delta$.*

Proof. Let $p = \gamma(t)$ be the point of γ that is farthest away from γ'. We first assume that t is at least 4δ away from the endpoints, namely $4\delta < t < L-4\delta$. Let $x = \gamma(t-4\delta), y = \gamma(t+4\delta)$ and x', y' be points on γ' that are closest to x, y respectively. Now consider the geodesic quadrilateral with vertices x, y, x', y'. Connect the diagonal between x and y' with a new geodesic to split the quadrilateral with two triangles. Now apply δ-hyperbolicity twice and get p is at most 2δ away from some point q on one of the edges $[x, x'], [x', y'], [y', y]$. If $q \in [x', y']$, then we are done since q is now a point on γ'. If $q \in [x, x']$, then use triangle inequality on the triangle $[x, p, q]$: the side $[x, p]$ has length 4δ by our choice, the side $[p, q]$ has length less than 2δ, so $[q, x]$ has length greater than 2δ. Now observe the path that goes from p to q then to x' has length strictly less than $[x, x']$, contradicting the maximality of our choice of p. The case where q lies on $[y, y']$ is similar. The case where t is less than 4δ away from one of the points follows from a similar argument. □

Lemma 1.14. *Let X be a δ-hyperbolic space. There does not exist an 8δ-local geodesic $\gamma : [0, L] \to X$ with $L \geq 8\delta$ and $\gamma(0) = \gamma(L)$.*

Proof. If such a local geodesic were to exist, we apply the above lemma and see γ lies in the disk of radius 2δ centered at $\gamma(0)$. However, since γ is 8δ-local geodesic, we know $d(\gamma(0), \gamma(8\delta)) = 8\delta$, a contradiction. □

Remark 1.15. In case X is a Cayley graph, if a path (local isometry $[a, b] \to X$) is not a c-local geodesic, then it must contains a segment of length $> c$ where if we go pass length c, the path stops being an isometry. This means there must be some other path of strictly shorter length that connects back to the satrting point. So we get a subword u, and a strictly shorter word v, such that the path corresponding to uv^{-1} form a loop, and u, v represent the same element.

Theorem 1.16. *Let G be a hyperbolic group with finite generating set S. Then G admits a Dehn presentation $G = \langle S | R \rangle$.*

Proof. Let $K > 8\delta$ be an integer and consider the set of reduced words in S of length at most K. We can check whether each pair of them represent the same elements. Since there are only finitely many S-words of length at most K, we get a finite set $\{(u_i, v_i)\}_{i=1}^m$ of pairs of words that represent the same element, where v_i has length strictly less than u_i. We claim $R := \{u_i v_i^{-1}\}$ gives us a Dehn presentation. The first two conditions of the definition for Dehn presentation are satisfied by construction.

Suppose w is reduced, non-empty and represents the identity. If its length does not exceed K, then we have $(w, \varnothing) \in R$ and w itself is one of the u_i's. If w has length greater than K, then following the path created by each letter of w on the Cayley graph, we get a loop of length at least $K > 8\delta$ at identity. By our previous lemma, this means w must not be a 8δ-local geodesic. By the remark we have a subword u of length at most $8\delta < K$ and some v of shorter length that represent the same element, which correspond to a sub-word of form u_i.

Lastly we need to show that R contains all necessary conditions to represent G. This follows from a similar argument as the last paragraph. □

Exercise 1.17. Show $\mathbb{R}^2$ is not hyperbolic.

Exercise 1.18. Prove $\mathbb{Z}_2 = \langle a, b | aba^{-1}b^{-1} \rangle$ is not a Dehn presentation.

Exercise 1.19. Give an example of a non-hyperbolic space that has c-local geodesic loops of arbitrarily large length and c.