STUDENT SEMINAR: QUASIMORPHISMS AND SYMPLECTIC GEOMETRY

Abstract. This document is written by the participants of the student seminar Quasimorphisms and Symplectic Geometry at ETH in the Autumn semester 2021. It provides a reference for the definitions and statements that are discussed during the seminar hours.

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Talk 0: Introduction

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Quasimorphisms are a tool to understand automorphism groups better. We start by giving definitions of different automorphism groups, state some well-known facts, then look at normal subgroups and define norms on these groups. The main reference for automorphism groups in this introduction is the book [Ban97] and the survey [Man21].

Let M be a connected smooth manifold throughout this introduction.

A. Automorphism groups

Let us first introduce several automorphism groups that we want to study. We can equip our manifold M with extra structure and study the group of isomorphisms that preserve this structure:

- $\mathrm{Homeo}(M)$, the group of homeomorphisms is defined for any topological space.
- $\mathrm{Diff}^r(M)$, the group of C^r-diffeomorphisms for $1 \leq r \leq \infty$. For $r = \infty$ we simply write $\mathrm{Diff}(M)$ for the group of smooth diffeomorphisms.
- $\mathrm{Diff}(M, \mathrm{vol})$, the group of diffeomorphisms that preserve a given volume form on M.
- $\mathrm{Iso}(M, g)$, the group of diffeomorphisms that preserve a given Riemannian metric on M.
- $\mathrm{Symp}(M, \omega)$, the group of diffeomorphisms that preserve a given symplectic form on M.
- $\mathrm{Ham}(M, \omega)$, the group of Hamiltonian diffeomorphisms of (M, ω).

When we put a topology on these groups one can do topology and for example try to find the homotopy type or the homotopy groups of these topological groups. For example $\mathrm{Diff}(S^1)$ is homotopy equivalent to $O(2)$. However, we are more interested in the algebraic structure in this seminar.

These groups are huge (except for $\mathrm{Iso}(M, g)$ which is compact for compact M). To mention one result of how large these groups are:

Theorem 0.1. *If* $\dim(M) \geq 2$*, then* $\mathrm{Diff}(M)$ *acts transitively on* k*-tuples of points in* M *for every* $k > 0$*.*

Moreover, many questions remain open, like:

Open question 0.2. *If* $\dim(M) \geq 2$*, is there a finitely generated, torsion-free group that is not isomorphic to some subgroup of* $\mathrm{Homeo}(M)$*?*

Another important question: Does the automorphism group determine the manifold and its structure?

Theorem 0.3 ([Whi63])**.** *Let* M *and* N *be compact manifolds both with or without boundary. If there is a group isomorphism* $\Phi : \mathrm{Homeo}(M) \to \mathrm{Homeo}(N)$ *then there is a homeomorphism* $w : M \to N$ *such that* $\Phi(\psi) = w\psi w^{-1}$ *for all* $\psi \in \mathrm{Homeo}(M)$*.*

Theorem 0.4 ([Fil82])**.** *Let* M *and* N *be manifolds without boundary. If there is a group isomorphism* $\Phi : \mathrm{Diff}^r(M) \to \mathrm{Diff}^s(N)$ *then* $r = s$ *and there is* C^r*-diffeomorphism* $w : M \to N$ *such that* $\Phi(\psi) = w\psi w^{-1}$ *for all* $\psi \in \mathrm{Diff}^r(M)$*.*

Similar statements hold for the volume-preserving and symplectic case (see [Ban97]).

B. Normal subgroups, simplicity and perfectness

Our main focus in this seminar lies on the algebraic analysis of automorphism groups. After some first thoughts we locate some proper normal subgroups:

- If M is non-compact one can restrict to compactly supported automorphisms (in this case we write subscripts $\mathrm{Diff}_c(M)$).
- If M has boundary one can restrict to automorphisms preserving a neighbourhood of the boundary (in this case we write subscripts $\mathrm{Diff}_{\partial M}(M)$). A topological group may be connected or not.

- The connected component that contains the identity is simple (we denote this component by $\mathrm{Diff}(M)_0$).

Recall that a group G is simple when it contains no non-trivial proper normal subgroup. For any group G the commutator subgroup $[G,G]$ is normal. If $G = [G,G]$ then we say G is *perfect*.

Theorem 0.5 (Anderson, Chernavski, Kirby, Edwards)**.** *For compact manifolds M the group*

$$\mathrm{Homeo}(M)_0$$

is perfect and simple.

Theorem 0.6 (Herman, Mather, Thurston, (A shorter proof: Mann [Man16]))**.** *The group*

$$\mathrm{Diff}_c^r(M)_0$$

is perfect and simple for $1 \leq r \leq \infty, r \neq n+1$, where n is the dimension of M.

The simplest case that is not covered by the theorem is still not settled.

Open question 0.7. *Is the group $\mathrm{Diff}^2(S^1)_0$ simple?*

In this seminar we will use the statement about the group of Hamiltonian diffeomorphisms:

Theorem 0.8 ([Ban78, Ban97])**.** *The group $\mathrm{Ham}(M)$ is perfect and simple for closed symplectic manifolds M.*

In the case of more structure we have the following results:

Theorem 0.9 (Thurston)**.** *Suppose M is closed and* vol *a volume form on M. Then the group*

$$[\widetilde{\mathrm{Diff}}(M,\mathrm{vol})_0, \widetilde{\mathrm{Diff}}(M,\mathrm{vol})_0]$$

is perfect and

$$[\mathrm{Diff}(M,\mathrm{vol})_0, \mathrm{Diff}(M,\mathrm{vol})_0]$$

is simple.

Theorem 0.10 ([Ban78])**.** *Suppose M is closed and ω a symplectic form on M. Then the group*

$$[\widetilde{\mathrm{Symp}}(M,\omega)_0, \widetilde{\mathrm{Symp}}(M,\omega)_0]$$

is perfect and

$$[\mathrm{Symp}(M,\omega)_0, \mathrm{Symp}(M,\omega)_0]$$

is simple.

For manifolds with boundary we have the following result:

Theorem 0.11 ([Fat80])**.** *The group*

$$\mathrm{Homeo}_{\partial B^n}(B^n, \mathrm{vol})$$

is simple for $n \geq 3$.

(Note that we omitted the subscript 0 as the group is arcwise connected due to the Alexander trick.) However, only very recently it was proved that:

Theorem 0.12 ([CGHS20])**.** *The group $\mathrm{Homeo}_{\partial B^2}(B^2, \mathrm{vol})$ is not simple.*

Open question 0.13. *Is the group $\mathrm{Homeo}(S^2, \mathrm{vol})_0$ simple?*

When we do not restrict to compactly support McDuff proved:

Theorem 0.14 ([McD80])**.** *The group $\mathrm{Diff}(\mathbb{R}^n)_0$ is perfect when $n \geq 3$.*

C. Fragmentation property

One important ingredient in the simplicity proofs is fragmentation. It is interesting itself as it leads to a norm on the automorphism groups, but usually it is very hard to compute explicitly.

Definition 0.15. A group of diffeomorphisms $G \subset \mathrm{Diff}(M)$ has the *fragmentation property* if for any open cover $\mathcal{U}$ of M and any $g \in G$, there are $g_1, \dots, g_s \in G$ with support $\mathrm{supp}(g_j) \subset U_j \in \mathcal{U}$ and $g = g_1 \cdots g_s$.

Theorem 0.16. *The group* $\mathrm{Homeo}_c(M)_0$ *and* $\mathrm{Diff}^r_c(M)_0$ *have the fragmentation property for all* $1 \leq r \leq \infty$.

D. Word norms

Suppose a group G is generated by a subset $\mathcal{S} \subset G$. That is, any element $g \in G$ can be written as a product of finitely many elements $s_1, \dots, s_n \in \mathcal{S}$: $g = s_1 \dots s_n$. Then there is a norm on G associated to $\mathcal{S}$, called the *word norm* $|| \cdot ||_{\mathcal{S}}$, defined by

$$||g||_{\mathcal{S}} = \inf\{n \in \mathbb{N} | \exists s_1, \dots, s_n \in \mathcal{S} : g = s_1 \dots s_n\}$$

Example 0.17. If G is perfect and simple, then $G = [G, G]$ is generated by the set of commutators: $\mathcal{S} = \{aba^{-1}b^{-1} | a, b \in G\}$. The corresponding word norm is called *commutator length* cl.

Example 0.18. If G has the fragmentation property, G is generated by the set

$$\mathcal{S} = \{g \in G \,|\, g \text{ is supported in an embedded ball}\}.$$

The corresponding word norm is called *fragmentation norm* frag.

Example 0.19. ([BKS18]) A diffeomorphism $\Psi \in \mathrm{Diff}_0(M)$ is called autonomous if there exists a vector field X with flow $\Psi_t \colon M \to M$ and $\Psi_1 = \Psi$. The set $\mathrm{Aut}(M)$ of all autonomous diffeomorphism generates $\mathrm{Diff}_0(M)$. The word norm associated with $\mathcal{S} = \mathrm{Aut}(M)$ is called the *autonomous norm*.

An interesting question is, whether $|| \cdot ||_{\mathcal{S}}$ is bounded or not. Here is a list of some known results:

Theorem 0.20 ([Ghy01, Theorem 6.2])**.** $cl(\mathrm{Homeo}_0(S^1)) = 1$.

Theorem 0.21 ([GG04])**.** *The commutator length* $cl(\mathrm{Diff}_0(S^2, \mathrm{vol}))$ *is unbounded.*

More advanced methods yield

Theorem 0.22 ([BIP08])**.** *The commutator length of the sphere* S^n *is uniformly bounded by*

$$\mathrm{cl}(\mathrm{Diff}_0(S^n)) \leq 4.$$

For any closed connected 3-dimensional manifold M,

$$\mathrm{cl}(\mathrm{Diff}_0(M)) \leq 10.$$

Theorem 0.23 ([Mil14])**.** *The commutator length on* $\mathrm{Diff}^r(A)_0$ *where* $r \neq 3$ *and* A *denotes the annulus is unbounded.*

Theorem 0.24. [BKS18, GG04] *For* $g \geq 1$ *the norm* aut *is unbounded on the group* $\mathrm{Ham}(\Sigma_g)$.

The proofs of the unboundedness statements involve quasimorphisms.

E. Quasimorphisms

Definition 0.25. A quasimorphism on a group G is a map

$$\mu \colon G \longrightarrow \mathbb{R}$$

for which there exists a constant $R \geq 0$ such that for all $g, h \in G$

$$|\mu(gh) - (\mu(g) + \mu(h))| \leq R.$$

The existence of unbounded quasimorphisms is not always easy. If they exist however, they can be of help to analyze the group.

Proposition 0.26. *Suppose G admits an unbounded quasimorphism μ that is bounded on $\mathcal{S}$. Then the word norm $||\cdot||_{\mathcal{S}}$ is unbounded.*

Proof. The quasimorphism property implies that for all $s_1, \ldots, s_n \in \mathcal{S}$

$$|\mu(s_1 \ldots s_n)| \leq nL$$

for some constant L. It follows that

$$||g||_{\mathcal{S}} \geq \frac{|\mu(g)|}{L}$$

and hence $||\cdot||_{\mathcal{S}}$ is unbounded whenever μ is unbounded. □

Any quasimorphism μ on G is bounded on commutators. Therefore, the proposition becomes simpler in case of the commutator length:

Corollary 0.27. *Suppose G is perfect and simple. If there exists an unbounded quasimorphism on G, then* $\mathrm{cl}\colon G \to \mathbb{N}$ *is unbounded.*

This motivates the construction of (unbounded) quasimorphisms! Here is a list of some existence results we will study in this seminar:

Theorem 0.28. [Ghy01] *There is a unique quasimorphism on* $\widetilde{\mathrm{Homeo}}_0(S^1)$.

Theorem 0.29. [GG04] *For any closed oriented surface Σ of genus at least 1, the vector space of unbounded quasimorphisms on* $\mathrm{Diff}_0(\Sigma, vol)$ *is infinite dimensional.*

Theorem 0.30. [Py06b, Py06a] *For any closed oriented surface Σ, the vector space of homogeneous Calabi quasimorphisms on* $\mathrm{Ham}(\Sigma)$ *is infinite dimensional.*

Theorem 0.31. [EP03] *There exists a continuous homogeneous Calabi quasimorphism on* $\mathrm{Ham}(S^2)$, $\mathrm{Ham}(S^2 \times S^2, \omega \oplus \omega)$ *and* $\mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$.

Open question 0.32. *Do there exist more than one continuous homogeneous Calabi quasimorphisms on* $\mathrm{Ham}(S^2)$*?*

Talk 1: Introduction to Quasimorphisms

Jonathan Clivio, Julian Huber

A. Homogeneous Quasimorphism

In this section we follow closely section 3.1 in [Car13].

Definition 1.1. For a quasimorphism $\mu : G \to \mathbb{R}$, we define the *defect* as

$$C := C(\mu) := \sup_{g,h \in G} |\mu(gh) - \mu(g) - \mu(h)|$$

For a fixed group G, we denote the set of quasimorphisms by $\mathrm{QM}(G)$. Note that $\mathrm{QM}(G)$ is a $\mathbb{R}$-vector space. One way how $\mu : G \to \mathbb{R}$ can trivially satisfy the quasimorphism inequality is if $\mu \in L^\infty(G)$. Hence, we can consider $L^\infty(G)$ as a subspace of limited interest. Hence, dividing out the "uninteresting" quasimorphisms, we come to the space $\mathrm{QM}(G)/L^\infty(G)$. The question whether there is a particular representative of this quotient classes leads us to this definition:

Definition 1.2. A quasimorphism $\mu : G \to \mathbb{R}$ is called *homogeneous* if the following holds:

$$\forall g \in G, \forall n \in \mathbb{Z} : \mu(g^n) = n\mu(g).$$

Or equivalently, for all $g \in G$, the map $\mu|_{\langle g \rangle} : \langle g \rangle \to \mathbb{R}$ is a group homomorphism.

We denote the set of homogeneous quasimorphisms by $\mathrm{QM}_h(G)$. It turns out homogeneous quasimorphisms are the representations, we were looking for:

Definition 1.3. For a quasimorphism $\mu : G \to \mathbb{R}$, we define the *homogenization* of μ by

$$\bar{\mu}(g) := \lim_{n \to \infty} \frac{\mu(g^n)}{n}.$$

To prove that this limit exists and defines a homogeneous quasimorphism we need this lemma

Lemma 1.4 (Fekete)**.** *Let $(a_n)_{n \geq 1}$ be a subadditive sequence of non-negative real numbers. By subadditive, we mean that for all $n, m \geq 1$, we have:*

$$a_{n+m} \leq a_n + a_m.$$

Then

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}.$$

Proposition 1.5. *Let $\mu : G \to \mathbb{R}$ be a quasimorphism. For each $g \in G$ the limit $\bar{\mu}(g)$ exists, the map $\bar{\mu} : G \to \mathbb{R}$ defines a homogeneous quasimorphism and $\mu - \bar{\mu} \in L^\infty(G)$. Moreover, for two homogeneous quasimorphisms μ' and μ'', we have $\mu' = \mu''$ or $\mu' - \mu'' \notin L^\infty(G)$.*

Proof. We first prove that the limit $\lim_{n\to\infty} \frac{\mu(g^n)}{n}$ exists. Fix $g \in G$. W.l.o.g., we may assume that $\mu(g) \geq 0$. We define the sequence $a_n := \mu(g^n) + (n+1)C$ where C is the defect of μ. We prove $a_n \geq n\mu(g) + 2C \geq 0$ by induction:

$$a_1 = \mu(g) + 2C \geq 0$$
$$a_{k+1} = a_k + \mu(g^{k+1}) - \mu(g^k) + C \geq a_k + \mu(g) = (k+1)\mu(g) + 2C$$

We used the quasimorphism property $|\mu(g^{k+1}) - \mu(g^k) - \mu(g)| \leq C$ as:

$$\mu(g^{k+1}) - \mu(g^k) - \mu(g) \geq -C$$
$$\implies \mu(g^{k+1}) - \mu(g^k) + C \geq \mu(g)$$

Further, we can use the quasimorphism property for any $k, l \geq 1$:

$$\mu(g^{k+l}) - \mu(g^k) - \mu(g^l) \leq C$$
$$\implies \mu(g^{k+l}) \leq \mu(g^k) + \mu(g^l) + C$$

Therefore, we find that (a_n) is a subadditive sequence:

$$a_{k+l} = \mu(g^{k+l}) + (k+l+C) \leq \mu(g^k) + \mu(g^l) + C + (k+l+1)C = a_k + a_l$$

We can hence apply the Lemma by Fekete:

$$\bar{\mu}(g) = \lim_{n\to\infty} \frac{\mu(g^n)}{n} = \lim_{n\to\infty} \frac{a_n - (n+1)C}{n} = \lim_{n\to\infty} \frac{a_n}{n} - C = \inf_n \frac{a_n}{n} - C$$

To prove $\mu - \bar{\mu} \in L^\infty(G)$, we compute:

$$|\mu(g^n) - n\mu(g)| \leq |\mu(g^n) - \mu(g^{n-1}) - \mu(g)| + \cdots + |\mu(g^2) - \mu(g) - \mu(g)| \leq (n-1)C$$

Therefore, we find:

$$|\mu(g) - \bar{\mu}(g)| = \lim_{n\to\infty} \frac{1}{n} |n\mu(g) - \mu(g^n)| \leq \lim_{n\to\infty} \frac{(n-1)C}{n} = C$$

Thus, we find also that $\bar{\mu}$ is a quasimorphism with defect $\leq 4C$:

$$\begin{aligned} |\bar{\mu}(gh) - \bar{\mu}(g) - \bar{\mu}(h)| &\leq |\bar{\mu}(gh) - \mu(gh)| + |\bar{\mu}(g) - \mu(g)| + \\ &+ |\bar{\mu}(h) - \mu(h)| + |\mu(gh) - \mu(g) - \mu(h)| \leq 4C \end{aligned}$$

To prove that $\bar{\mu}$ is homogeneous, we compute:

$$|\bar{\mu}(g^n) - n\bar{\mu}(g)| = \lim_{k\to\infty} \frac{1}{k} |\bar{\mu}(g^{kn}) - n\mu(g^k)| \leq \lim_{k\to\infty} \frac{(n-1)}{k} C = 0$$

Now, let μ' and μ'' be homogeneous quasimorphism. Assume that $|\mu'(g) - \mu''(g)| =: \varepsilon > 0$. Then we find:

$$\lim_{n\to\infty} |\mu'(g^n) - \mu''(g^n)| = \lim_{n\to\infty} n\varepsilon = \infty$$

□

Another perspective is that this result gives us the following isomorphism.

Corollary 1.6. *For any group G:*

$$\mathrm{QM}(G)/L^\infty(G) \cong \mathrm{QM}_h(G)$$

A nice fact about homogeneous quasimorphism is the following.

Proposition 1.7. *Let μ be a homogeneous quasimorphism. Then μ is a class function i.e. for $g, h \in G$, we have:*

$$\mu({}^h g) = \mu(g),$$

where use the notation ${}^h g := hgh^{-1}$.

Proof. We compute:

$$\begin{aligned} \mu({}^h g^n) &= \mu(({}^h g)^n) = n\mu({}^h g) \\ \mu(g^n) &= n\mu(g) \\ \mu(h^{-1}) &= -\mu(h) \end{aligned}$$

For any n, we therefore find:

$$\begin{aligned} |\mu({}^h g) - \mu(g)| &= \frac{1}{n} |\mu({}^h g^n) - \mu(g^n)| \\ &= \frac{1}{n} |\mu({}^h g^n) - \mu(h) - \mu(h^{-1}) - \mu(g^n)| \\ &\leq \frac{1}{n} |\mu({}^h g^n) - \mu(h) - \mu(g^n h^{-1})| + \frac{1}{n} |\mu(g^n h^{-1}) - \mu(g^n) - \mu(h^{-1})| \\ &\leq \frac{2C}{n} \end{aligned}$$

Taking the limit with respect to n on both sides, we find the desired result. □

Proposition 1.8. *Let μ be a homogeneous quasimorphism and let $g, h \in G$ be such that $gh = hg$. Then it follows that $\mu(gh) = \mu(g) + \mu(h)$.*

Proof. Denote by C the defect of μ. Using the fact that μ is homogeneous we obtain that

$$\begin{aligned}|\mu(gh) - \mu(g) - \mu(h)| &= |\frac{1}{n}\mu((gh)^n) - \frac{1}{n}\mu(g^n) - \frac{1}{n}\mu(h^n)| \\ &= \frac{1}{n}|\mu(g^n h^n) - \mu(g^n) - \mu(h^n)| \\ &\leq \frac{C}{n}.\end{aligned}$$

By sending $n \to \infty$ it follows that $|\mu(gh) - \mu(g) - \mu(h)| = 0$, which completes the proof. □

B. Simple and perfect groups

In this section we follow section 4.1 in [Pin20] and [Bar].
Let G be any group.

Definition 1.9. Suppose $a, b \in G$ then the commutator of a and b is defined by

$$[a, b] := aba^{-1}b^{-1}.$$

Definition 1.10. The commutator group of G is defined by

$$[G, G] := \langle\{[a, b] : a, b \in G\}\rangle,$$

i.e. the subgroup generated by all commutators of G.

Definition 1.11. Let $g \in [G, G]$. The commutator length of g is defined by

$$comm(g) := min\left\{k \in \mathbb{N} \ : \ g = \prod_{i=1}^{k}[a_i, b_i], \ a_i, b_i \in G\right\}.$$

Definition 1.12. A group G is called perfect if $G = [G, G]$.

Example 1.13 (Rose)**.** The special linear group $SL_2(\mathbb{F}_{p^r})$ with p prime number, $r \geq 1$ and $p^r > 3$ is perfect.

Definition 1.14. A non trivial group G is called simple if the trivial group $\{1_G\}$ and G are the only normal subgroups of G.

Example 1.15.

- For $n \geq 5$ the alternating group A_n is simple .
- For $n \geq 2$ and any field F the group

$$PSL(n, F) := SL_n(F)/\{\lambda I_n \mid \lambda \in F^\times, \lambda^n = 1\}$$

  is simple except in the case where $n = 2$ and $|F| \leq 3$.

Proposition 1.16. *Suppose G is simple and non-abelian then it follows that G is perfect.*

Proof. Note that since G is non-abelian and not trivial, $[G, G]$ is not trivial. It follows directly from the definition that $[G, G]$ is normal in G. So G being simple implies that $G = [G, G]$, i.e. G is perfect. □

Remark 1.17. Note that the converse of Proposition 1.16 not true in general. A counterexample would be $G := SL_2(\mathbb{F}_{p^r})$ with $p \geq 3$ prime number and $r \geq 1$. The group G is perfect as seen in Example 1.13. Note that the center of any group is always normal and that the center of G consists of the elements I_2 and $-I_2$ which are distinct because the characteristic of our field $\mathbb{F}_{p^r}$ is not 2. So the group center is not trivial. This shows that G is not simple.

C. Quasimorphisms and commutator length

The following proposition yields a strategy to prove unboundedness of the commutator length. It is almost the same as Proposition 0.26 and the following version is taken from [GG04].

Proposition 1.18. *Let $\Phi\colon G \to \mathbb{R}$ be a non-trivial homogeneous quasimorphism then commutator length $comm\colon [G, G] \to \mathbb{N}$ is an unbounded function.*

Proof. Denote by C the defect of Φ. Since by assumption Φ is non-trivial, $C > 0$. Let $g \in [G, G]$ be arbitrary. Then we can write $g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$ with all $a_j,\ b_j \in G$ and $k = comm(g)$. Using the triangular inequality and the properties of homogeneous quasimorphisms we obtain the following estimate:

$$\begin{aligned}|\Phi(g)| &= |\Phi(g) - (\Phi(a_1) + \Phi(b_1) - \Phi(a_1) - \Phi(b_1) + \dots)|\\ &= |\Phi(g) - (\Phi(a_1) + \Phi(b_1) + \Phi(a_1^{-1}) + \Phi(b_1^{-1}) + \dots)|\\ &\leq (4k-1)C\end{aligned}$$

So we obtain that

$$comm(g) = k \geq \frac{1}{4}\left(\frac{|\Phi(g)|}{C} + 1\right)$$

which implies that

$$comm(g) \geq \frac{|\Phi(g)|}{4C}.$$

Using this inequality with g^p and the fact that Φ is homogeneous we obtain

$$comm(g^p) \geq p\frac{|\Phi(g)|}{4C}.$$

This shows that *comm* is an unbounded function. □

D. Examples of Automorphism groups

In this section we explain the tilde-notation that appears in theorems 0.9 and 0.10 for this we follow [Row].

Definition 1.19. A *universal cover* of a connected topological space X is a simply connected space $\widetilde{X}$ with a covering map $f : \widetilde{X} \to X$.

Proposition 1.20. *A universal cover exists if and only if X is connected, locally path-connected and semilocally simply connected.*

In this case we can explicitly construct the universal cover: let X be connected, locally path-connected and semilocally simply connected with $x_0 \in X$ define

$$Y := \{\text{paths starting at } x_0\} = \{\gamma : [0,1] \to X \mid f(0) = x_0\}.$$

We define the following equivalence relation on Y:

$$\gamma \sim \gamma' :\Longleftrightarrow \gamma(1) = \gamma'(1) \text{ and } \gamma, \gamma' \text{ are homotopy equivalent with respect to } \{0,1\}.$$

Then we set $\widetilde{X} := Y/\sim$ with the map

$$\begin{aligned} f : \widetilde{X} &\to X\\ [\gamma]_\sim &\mapsto \gamma(1).\end{aligned}$$

In the explicit example of a connected automorphism group G of an object X, we can choose the identity $\mathrm{id}_X \in G$ as base-point. Further we define the set

$$Y := \{f : [0,1] \times X \to X \mid f(0,x) = x \wedge \forall t \in [0,1] : f(t,\cdot) \in G\}$$

and the equivalence relation

$$\begin{aligned} f \sim g :\Longleftrightarrow\ & f(1,x) = g(1,x) \wedge \exists F : [0,1] \times [0,1] \times X \to X \text{ continuous with}\\ & F(0,t,x) = f(t,x) \wedge F(1,t,x) = g(t,x) \wedge F(s,0,x) = x \wedge F(s,1,x) = f(1,x)\end{aligned}$$

This construction explains the notation used in the theorems 0.9 and 0.10.

Talk 2: The Homeomorphism group of the circle

Yilin Ni, Reto Kaufmann

The goal of today's talk is to show that the group of homeomorphisms of the circle $\mathrm{Homeo}_0(S^1)$ is simple and perfect. To do so, we are first going to exhibit that any topological group, hence in particular $\mathrm{Homeo}_0(M)$ for any manifold M, is generated by a neighbourhood of the identity. Next we are going to define the fragmentation property and show that for any $\mathrm{Homeo}_0(M)$ this property implies perfectness and simplicity. Finally we will conclude by showing that $\mathrm{Homeo}_0(S^1)$ indeed has the fragmentation property.

A. Topological Groups

Definition 2.1. A *topological group* G is a group endowed with a topology with respect to which the group operations are continuous.

Definition 2.2. A neighbourhood U of the identity $e \in G$ is *symmetric* if

$$g \in U \Leftrightarrow g^{-1} \in U$$

or equivalently if $U = U^{-1}$.

Remark 2.3. Any neighbourhood V of the identity contains a symmetric neighbourhood. Concretely we might take

$$U = V \cap V^{-1}$$

which clearly satisfies $U = U^{-1}$.

Lemma 2.4. *Every open subgroup of a topological group is also closed.*

Proof. Let $H < G$ be an open subgroup. Since left translation L_g is a homeomorphism, $L_g(H) = gH$ is open. It follows that $G\backslash H$ is open as well since its the union of such open sets

$$G\backslash H = \bigcup_{g \in G\backslash H} gH$$

Hence H is closed. □

Proposition 2.5. *Any connected topological group G is generated by any neighbourhood V of the identity i.e.*

$$G = \bigcup_{n=1}^{\infty} V^n$$

Proof. Let $U \subset V$ be a symmetric neighbourhood of the identity. Then

$$H = \bigcup_{n=1}^{\infty} U^n$$

is an open subgroup. By the Lemma it is also closed and since G is connected it follows that $H = G$. □

B. The Fragmentation Property

The standard reference for the fragmentation property of various automorphism groups is [Ban97].

Definition 2.6. The *support* of a homeomorphism $f : M \to M$ is the closure of the points which are not fixed by f, i.e.

$$\mathrm{supp}(f) = \overline{\{x \in M : f(x) \neq x\}}$$

Definition 2.7. A group $G \subset \mathrm{Homeo}_0(M)$ has the *fragmentation property* if for any finite open cover $\mathcal{U}$ of M and any element $g \in G$, there is a decomposition

$$g = g_1 \circ g_2 \circ \cdots \circ g_n$$

with $g_i \in G$ and where the support of each g_i is contained in some set of $\mathcal{U}$.

The following proposition is due to Anderson [And58]. We follow the exposition given in [Man15].

Proposition 2.8. *If $Homeo_0(M)$ has the fragmentation property, then it is perfect.*

Proof. If $\mathrm{Homeo}_0(M)$ has the fragmentation property, we can reduce to the case of considering homeomorphisms whose support is contained in some small open ball B.

Let thus B be such an open ball in M and $f \in \mathrm{Homeo}_0(M)$ such that $\mathrm{supp}(f) \subset B$. We can then choose $b \in \mathrm{Homeo}_0(M)$ such that $b^n(B) \cap b^m(B) = \emptyset$ for $n \neq m$. To see that this is indeed possible, consider a chart containing the open ball B and identify the codomain with all of $\mathbb{R}^n$. b can then be taken to be any translation by at least the diameter of the ball.

Define then
$$a(x) = \begin{cases} b^n f b^{-n}(x) & \text{if } x \in b^n(B) \text{ for } n > 0 \\ x & \text{otherwise} \end{cases}$$
and observe that $a \in \mathrm{Homeo}_0(M)$ is simply a copy of f in each $b^n(B)$ for $n > 0$ and the identity in B as well as outside all balls. Note also that a^{-1} is exactly the same for f^{-1}.

Finally, we consider the commutator of a with b
$$[a, b] = a^{-1}b^{-1}ab.$$
Outside of all $b^n(B)$ the action of this commutator is the identity so we focus only on the ball B and its copies $b^n(B)$. There it acts as the composition of the following actions:

(1) Every ball $b^n(B)$ gets sent to its successor $b^{n+1}(B)$.
(2) We apply a, that is, we apply f in all balls $b^n(B)$ except B.
(3) Every ball $b^{n+1}(B)$ gets sent back to its predecessor.
(4) We apply a^{-1}, that is, we apply f^{-1} in all balls except B.

There are now two important cases to consider:

$n \geq 1$: The action of a in $b^{n+1}(B)$ is f and the action of a in $b^n(B)$ is f^{-1} so that the net action of the commutator on $x \in b^n(B)$ is $x \mapsto f^{-1} \circ f(x) = x$.

$n = 0$: The action of a in $b^1(B)$ is f and the action of a in $b^n(B)$ is the identity so that the net action of the commutator on $x \in B$ is $x \mapsto f(x)$.

Hence we conclude that
$$f = [a, b]$$
which finishes the proof. □

Remark 2.9. Inspecting the previous proof we have shown that any element in $\mathrm{Homeo}_c(\mathbb{R}^n)$ can be written as a commutator, that is $\mathrm{cl}(\mathrm{Homeo}_c(\mathbb{R}^n)) = 1$.

Remark 2.10. Note that we do not require the full strength of the fragmentation property. For this proof it is sufficient to require that *there exists* an open cover $\mathcal{U}$ such that any homeomorphism can be decomposed into a product of elements with support in an open set of that open cover. It is this weaker form of the fragmentation property that we are going to prove for S^1.

The following discussion focuses on the relative simple case $\mathrm{Homeo}_0(S^1)$. We recall the definition:

Definition 2.11. Let M be a connected compact manifold. Then $\mathrm{Homeo}_0(M)$ is defined as the path-component of $\mathrm{Homeo}_0(M)$ containing the identity. We further assume M is orientable, then define $\mathrm{Homeo}_+(M)$ to be the group of all orientation preserving homeomorphisms.

Proposition 2.12. *If $M = S^n$, then* $\mathrm{Homeo}_+(M) = \mathrm{Homeo}_0(M)$.

Proof. We use the following theorem without proof:

Theorem 2.13 (Hopf)**.** *Let M be a connected oriented compact n-manifold. Then two continuous maps $f, g : M \to S^n$ are homotopic if and only if f and g have the same degree.*

If $f \in \mathrm{Homeo}(S^n)$, then it can only have degree 1 or -1, since
$$1 = \deg(\mathrm{id}) = \deg(f \circ f^{-1}) = \deg(f)\deg(f^{-1}).$$
Therefore $\mathrm{Homeo}(S^n)$ consists of exactly 2 path-components. The groups of orientation-preserving maps are exactly the path-component containing the identity. □

Then we turn to prove the (weak) fragmentation property of S^1. The following intuition may be useful: the compact-open topology on $\mathrm{Homeo}_0(S^1)$ agrees with the topology of uniform convergence, hence a map is close to identity if it doesn't move any point far.

Proposition 2.14 (weak fragmentation property of S^1)**.** *Let $n \geq 3$, $\mathcal{U} = \{I_1, I_2, \ldots, I_n\}$ a family of successive open intervals (balls) covering S^1 such that the intersection of neighbouring intervals is nonempty, and the intersection of any three intervals is empty. Then* $\mathrm{Homeo}_0(g)$ *has fragmentation with respect to $\mathcal{U}$.*

Proof. We prove the case for $n = 3$. The proof can be easily generalized. Note that this is already sufficient for the proof of perfectness and simplicity.

Let G_1, G_2, G_3 be subgroups of $\mathrm{Homeo}_0(S^1)$ supported by I_1, I_2, I_3, respectively. Since $\mathrm{Homeo}_0(S^1)$ is generated by a neighborhood of identity, it suffices to find a neighborhood U of identity such that all maps in U can be decomposed into element in G_1, G_2, G_3. Let $x_{1,2}, x_{2,3}, x_{3,1}$ be points in the interior of $I_i \cap I_j$. Let $U \in \mathrm{Homeo}_0(S^1)$ be a neighborhood of identity such that for all $f \in U$, $f(x_{i,j}) \in I_i \cap I_j$. Let $f \in U$. Let g_{12}, g_{23}, g_{31} be maps supported by $I_i \cap I_j$ and agree with f on a neighborhood of x_{ij}.

Then $h = g_{12}^{-1} g_{23}^{-1} g_{31}^{-1} f$ fixes a neighbourhood of x_{ij}. This however implies h can be decomposed into elements in G_1, G_2 and G_3. □

Remark 2.15. Does the property hold for $n = 2$? We may need the orientation preserving property. Does the weak fragmentation property hold for S^n in general (given possibly stronger assumptions)?

Corollary 2.16. $\mathrm{Homeo}_0(S^1)$ *is simple.*

Proof. Let $n \neq \mathrm{id}$ be an element of $\mathrm{Homeo}_0(S^1)$. Then there exists an interval $I \in B$ such that $n(I) \cap I = \emptyset$. We would like to show that all $f \in \mathrm{Homeo}_0(S^1)$ is in the normal closure of n. First suppose $\mathrm{supp}(f) \subset I$, then as seen in proposition 2.8, we can write $f = [a, b]$ for some a, b supported by I (slightly modify the proof, take some proper sub-interval $I' \subset I$ and make $b^n(I') \subset I$).

Now let $g \in \mathrm{Homeo}_0(S^1)$ be such that $g \mid_I = \mathrm{id}$ and $g(n(I)) \cap n(I) = \emptyset$. Then

$$f = [[a, n], [b, gng^{-1}]].$$

Now if the support of f is contained in some other interval $\hat{I}$, we can construct some map h such that $h(\hat{I}) \subset I$. Then $\mathrm{supp}(hfh^{-1})$ is contained in I. □

Remark 2.17. The proof is in fact valid for a any manifold that satisfies the fragmentation property.

Talk 3: Rotation and Translation Numbers

Arthur, Moritz

A. The Universal Cover $\widetilde{\text{Homeo}_0}(S^1)$

Define $S^1 := \mathbb{R}/\mathbb{Z}$ and let $\text{Homeo}(S^1)$ denote the set of all homeomorphisms of S^1. Then $d_\infty : (f,g) \mapsto \sup_{x\in S^1} d(f(x), g(x))$ endows $\text{Homeo}(S^1)$ with a metric structure. Recall that the metric space $(\text{Homeo}(S^1), d_\infty)$ has exactly two arcwise connected components. The connected component containing the identity function id, which shall be denoted by $\text{Homeo}_0(S^1)$, coincides with the set of all orientation preserving homeomorphisms of S^1.

Remark 3.1. The metric space $(\text{Homeo}_0(S^1), d_\infty)$ is not simply connected.

Proof. Define for all $t \in [0,1]$, $\omega_t : S^1 \to S^1, \bar{x} \mapsto \bar{x} + \bar{t}$, the rotation by t, and let $\gamma : [0,1] \to \text{Homeo}_0(S^1), t \mapsto \omega_t$. Since $\gamma(0) = \gamma(1) = \text{id}$ and the ω_t are all orientation preserving, γ is a loop in $\text{Homeo}_0(S^1)$.

Suppose now that there exists a homotopy H from γ to id relative to $\{0,1\}$. Then for all $s \in [0,1]$, $H(0,s)(0) = H(1,s)(0) = id(0) = 0$. Hence $\bar{H} : (x,s) \mapsto H(x,s)(0)$ is a homotopy rel.$\{0,1\}$ from $\alpha : t \mapsto H(t,1)(0) = \gamma(t)(0) = \bar{t}$ to 0. This is a contradiction. □

Henceforth, we shall denote the quotient map from $\mathbb{R}$ to S^1 by p. Recall that $\mathbb{R} \xrightarrow{p} S^1$ is the universal cover of S^1. Furthermore, τ_1 shall denote the translation by 1.

Definition 3.2. Let $\tilde{f} \in \text{Homeo}(\mathbb{R})$. We say that $\tilde{f}$ commutes with integral translations if for all $x \in \mathbb{R}$, $\tilde{f}(x+1) = \tilde{f}(x) + 1$, i.e. $\tau_1 \tilde{f} = \tilde{f}\tau_1$. The set of all $\tilde{f} \in \text{Homeo}(\mathbb{R})$ that commute with integral translations will be denoted by $\widetilde{\text{Homeo}_0}(S^1)$.

Take $\tilde{f} \in \widetilde{\text{Homeo}_0}(S^1)$. Then $\tilde{f} - \text{id}$ is continuous and 1-periodic. Hence, in particular $\tilde{f} - \text{id}$ is bounded. It follows that for any $\tilde{f}, \tilde{g} \in \widetilde{\text{Homeo}_0}(S^1)$, $\tilde{f} - \tilde{g}$ is bounded. Thus $d_\infty : (f,g) \mapsto \sup_{x\in\mathbb{R}} d(\tilde{f}(x), \tilde{g}(x))$ is well-defined and endows $\widetilde{\text{Homeo}_0}(S^1)$ with a metric structure.

One can easily check that the composition of functions endows $\widetilde{\text{Homeo}_0}(S^1)$ with a topological group structure. In fact, take $\tilde{f} \in \widetilde{\text{Homeo}_0}(S^1)$ and $x \in \mathbb{R}$. Then, $\tilde{f}(\tilde{f}^{-1}(x+1)) = x + 1 = \tilde{f}(\tilde{f}^{-1}(x)) + 1$. Now $\tilde{f}$ is bijective and we know that $\tilde{f}(\tilde{f}^{-1}(x)) + 1 = \tilde{f}(\tilde{f}^{-1}(x) + 1)$. Thus, $\tilde{f}^{-1}(x) + 1 = \tilde{f}^{-1}(x+1)$.

Lemma 3.3. *Let $\tilde{f} \in \widetilde{\text{Homeo}_0}(S^1)$. Then $\tilde{f}$ induces a homeomorphism $\pi(\tilde{f}) \in \text{Homeo}_0(S^1)$. Furthermore, for all $x, x' \in \mathbb{R}$, with $|x - x'| \leq 1$, we have $|(\tilde{f}(x) - x) - (\tilde{f}(x') - x')| \leq 1$.*

Proof. Let $\tilde{f} \in \widetilde{\text{Homeo}_0}(S^1)$ and take $x, y \in \mathbb{R}$ such that there exists $k \in \mathbb{Z}$ such that $x - y \in \mathbb{Z}$. Then $p \circ \tilde{f}(x) = p \circ \tilde{f}(y)$. Hence, $p \circ \tilde{f}$ induces a continuous map $\pi(\tilde{f})$ from S^1 to S^1. Since $\pi(\text{id}_\mathbb{R}) = \text{id}_{S^1}$, $\pi(\tilde{f}^{-1}) = \pi(\tilde{f})^{-1}$, hence $\pi(\tilde{f})$ is homeomorphism.

For the second point, we first notice that $\tilde{f}$ is strictly increasing. Take $x, x' \in \mathbb{R}$, with $|x - x'| \leq 1$, w.l.o.g $x' > x$. Then, $x < x' < x + 1$. Hence, $\tilde{f}(x) + 1 > \tilde{f}(x') > \tilde{f}(x)$, thus $0 < \tilde{f}(x') - \tilde{f}(x) < 1$. Hence, $0 < \tilde{f}(x') - x + x - \tilde{f}(x) < 1$. Finally $0 < x < x' < 1$, thus, $-1 < \tilde{f}(x') - x' + x - \tilde{f}(x) < 1$ and the proof is complete. □

Next we will see that the map π is actually the covering map of the universal cover of $\text{Homeo}_0(S^1)$.

Proposition 3.4. *The metric space $\widetilde{\text{Homeo}_0}(S^1)$ endowed with d_∞ is simply connected. Furthermore, the map*

$$\begin{aligned} \pi : \widetilde{\text{Homeo}_0}(S^1) &\to \text{Homeo}_0(S^1) \\ \tilde{f} &\mapsto (p(x) \mapsto p(\tilde{f}(x))) \end{aligned} \tag{1}$$

is a group homomorphism and $\widetilde{\text{Homeo}_0}(S^1) \xrightarrow{\pi} \text{Homeo}_0(S^1)$ is the universal cover of $\text{Homeo}_0(S^1)$.

Proof. The first point is a direct consequence from the fact that $\widetilde{\mathrm{Homeo}_0}(S^1)$ is a convex subset of the vector space $\mathbb{R}^{\mathbb{R}}$ and the continuity of the addition and the scalar multiplication with respect to d_∞ on $\widetilde{\mathrm{Homeo}_0}(S^1)$.

Explicitly, take $\gamma : [0,1] \to \widetilde{\mathrm{Homeo}_0}(S^1)$ a loop based at id. Then for all $t, \lambda \in [0,1]$, $\lambda \tilde{f} + (1-\lambda)\,\mathrm{id}$ is a bijective and continuous map from $\mathbb{R}$ to $\mathbb{R}$ and thus a homeomorphism. Hence, it is easy to see that $\lambda \tilde{f} + (1-\lambda)\,\mathrm{id} \in \widetilde{\mathrm{Homeo}_0}(S^1)$ and we may define the homotopy

$$\begin{aligned} H : [0,1] \times [0,1] &\to \widetilde{\mathrm{Homeo}_0}(S^1) \\ (t,s) &\mapsto s\gamma(t) + (1-s)\,\mathrm{id}\,. \end{aligned} \tag{2}$$

from $\tilde{f}$ to id relative to $\{0,1\}$. Hence, $\widetilde{\mathrm{Homeo}_0}(S^1)$ is simply connected.

Now it remains to prove that π is indeed a covering map. Let us start with the most delicate part, the surjectivity of π.

Take $f \in \mathrm{Homeo}_0(S^1)$. Then $f \circ p$ is a continuous map from $\mathbb{R}$ to S^1. Now, since $\mathbb{R} \xrightarrow{p} S^1$ is the universal cover and $\mathbb{R}$ is simply connected, $f \circ p$ can be lifted to a map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ i.e. there exists a map $\tilde{f}$ that satisfies $p \circ \tilde{f} = f \circ p$.

Let $\tilde{g}$ be another lift of f. Since $p \circ \tilde{f} = p \circ \tilde{g}$, $\mathrm{Im}(\tilde{f} - \tilde{g}) \subset \mathbb{Z}$ and by continuity of $\tilde{f} - \tilde{g}$, we conclude that there exists $k \in \mathbb{Z}$ such that $\tilde{f} = \tilde{g} + k$.

Next we shall prove that $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(S^1)$. Let $g \in \mathrm{Homeo}_0(S^1)$. The above shows that there exists a lift $\tilde{g} : \mathbb{R} \to \mathbb{R}$ of g. Since

$$p \circ \tilde{g} \circ \tilde{f} = g \circ p \circ \tilde{f} = g \circ f \circ p, \tag{3}$$

$\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f$. In particular, if we choose $g = f^{-1}$, we get that $\tilde{g} \circ \tilde{f}$ is a lift of the identity. Since $\mathrm{id}_{\mathbb{R}}$ is obviously another lift of $\mathrm{id}\, S^1$, there exists $k \in \mathbb{Z}$ such that $\tilde{g} \circ \tilde{f} = \mathrm{id} + k$. Hence, $\tilde{g} - k = \tilde{f}^{-1}$ and $\tilde{f}$ is a homeomorphism and increasing since f is orientation preserving. Now, $\tilde{f} \circ \tau_1$ is another lift of f, hence for all $x \in \mathbb{R}$, $\tilde{f}(x+1) = \tilde{f}(x) + 1$. Thus, $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(S^1)$.

Finally, by construction of π, it is easy to see that $\pi(\tilde{f}) = f$ for every lift $\tilde{f}$ of f. We conclude that π is surjective. A posteriori, (3) yields that π is a group morphism.

Now it just remains to prove that there exists an open cover $(U_i)_{i \in I}$ of S^1 such that for every $i \in I$, $\pi^{-1}(U_i)$ is a union of disjoint open sets $(V_{i,j})_{j \in J}$ and such that $\pi : V_{i,j} \to U_i$ is a homeomorphism for all $j \in J$.

This follows from the following observation: take $f, g \in \widetilde{\mathrm{Homeo}_0}(S^1)$ such that $d_\infty(f,g) < \frac{1}{2}$. Then for all $x \in \mathbb{R}$, $d(f(x), g(x)) = d(\pi(f)(x), \pi(g)(x))$. Hence $d_\infty(f,g) = d_\infty(\pi(f), \pi(g))$. Thus choose $f \in \widetilde{\mathrm{Homeo}_0}(S^1)$, let $\tilde{f}$ be a lift of f and set $U := B_{d_\infty}(f, \frac{1}{3})$. Then $\pi^{-1}(U)$ is a disjoint union of open sets, namely $\bigsqcup_{n \in \mathbb{Z}} (B_{d_\infty}(\tilde{f}, \frac{1}{3}) + n)$. Finally, take $k \in \mathbb{Z}$. By the observation above, $\pi : B_{d_\infty}(\tilde{f}, \frac{1}{3}) + k \to B_{d_\infty}(f, \frac{1}{3})$ is a surjective isometry and hence a homeomorphism. Therefore, $\widetilde{\mathrm{Homeo}_0}(S^1) \xrightarrow{\pi} \mathrm{Homeo}_0(S^1)$ is the universal cover of $\mathrm{Homeo}_0(S^1)$. □

B. Translation and Rotation Numbers

We recall that $\mathrm{Homeo}_0(S^1) = \mathrm{Homeo}_+(S^1)$. Let $\tilde{f} \in \widetilde{\mathrm{Homeo}_+}(S^1)$, *i.e.* a homeomorphism of $\mathbb{R}$ which commutes with integral translations. If two points in $\mathbb{R}$ differ by at most 1, so do their images by $\tilde{f}$.

$$|(\tilde{f}(x) - x) - (\tilde{f}(x') - x')| \leq 1. \tag{4}$$

We define

$$T(\tilde{f}) = \tilde{f}(0) = \tilde{f}(0) - 0. \tag{5}$$

T is a quasimorphism, because, given $\tilde{f}_1, \tilde{f}_2 \in \widetilde{\mathrm{Homeo}_+}(S^1)$, we see that

$$\left|\tilde{f}_1(\tilde{f}_2(0)) - \tilde{f}_2(0) - \tilde{f}_1(0)\right| = \left|\Big(\tilde{f}_1(\tilde{f}_2(0)) - \tilde{f}_2(0)\Big) - \Big(\tilde{f}_1(0) - 0\Big)\right| \leq 1$$

by Equation (4).

Definition 3.5 (Translation Number)**.** The *translation number* $\tau(\tilde{f})$ is defined as the homogenisation of $T(\tilde{f})$.

Remark 3.6. We could also have defined $T(\tilde{f})$ as $\tilde{f}(x)-x$ for an arbitrary $x \in [0,1]$. Since $\tilde{f}(0)-0$ and $\tilde{f}(x)-x$ differ by at most 1 by Lemma 3.3 and $\mathrm{QM}(G)/L^\infty(G) \cong \mathrm{QM}_h(G)$ by Proposition 1.5, both quasi-morphisms have the same homogenisation i.e. the translation number does not depend on the choice of $x \in [0,1]$.

Example 3.7. Consider the translation by a number $k \in \mathbb{R}$: $\tilde{f} = x + k$. Then $T(\tilde{f}) = \tilde{f}(0) - 0 = 0 + k = k$. The translation number evaluates to

$$\tau(\tilde{f}) = \lim_{n\to\infty} \frac{T(\tilde{f}^n)}{n} = \lim_{n\to\infty} \frac{nk}{n} = k. \tag{6}$$

Proposition 3.8. *The translation number τ is a homogeneous quasimorphism from $\widetilde{\mathrm{Homeo}}_+(S^1)$ to $\mathbb{R}$.*

Proof. Since the translation number has been defined as the homogenisation of T, the result follows from Proposition 1.5. □

We will show that τ is the unique non-trivial homogeneous quasimorphism $\tau : \widetilde{\mathrm{Homeo}}_+(S^1) \longrightarrow \mathbb{R}$. For this, we need the following lemmata: 9

Lemma 3.9. *Let $f \in \mathrm{Homeo}_0(S^1)$. Then there exist $f_i \in \mathrm{Homeo}_0(S^1)$ such that $f = [f_1, f_2][f_3, f_4]$ i.e. f is a product of two commutators.*

Proof. Take $f \in \mathrm{Homeo}_0(S^1)$. Then there exist $g, h \in \mathrm{Homeo}_0(S^1)$ such that $\mathrm{supp}(g)$ and $\mathrm{supp}(h)$ are strictly contained in S^1 and $f = gh$. Hence, the result follows from Remark 2.9. □

Proposition 3.10. *The translation number is the unique homogeneous quasimorphism $\tau : \widetilde{\mathrm{Homeo}}_+(S^1) \longrightarrow \mathbb{R}$ which takes the value 1 on the translation by 1.*

Proof. Suppose there exist two such quasimorphisms φ and ψ. Take $\tilde{f} \in \widetilde{\mathrm{Homeo}_0}(S^1)$. We know that $\tilde{f}$ commutes with integral translations, i.e. $\tilde{f} \circ \tau_1 = \tau_1 \circ \tilde{f}$. Now, since φ and ψ are homogeneous they act on the group generated by ta_1 and $\tilde{f}$ as homomorphisms. In particular, for all $n \in \mathbb{Z}$,

$$\begin{aligned}(\varphi - \psi)(\tau_1^n \tilde{f}) &= n(\varphi(\tau_1) - \psi(\tau_1)) + (\varphi - \psi)(\tilde{f}) \\ &= (\varphi - \psi)(\tilde{f})\end{aligned} \tag{7}$$

since $\varphi(\tau_1) = \psi(\tau_1) = 1$. Hence, for all $\tilde{g} \in \pi^{-1}(\{\pi(\tilde{f})\})$, $(\varphi - \psi)(\tilde{f}) = (\varphi - \psi)(\tilde{g})$. Since π is a group morphism, $\varphi - \psi$ induces thus a homogeneous quasimorphism on $\mathrm{Homeo}_0(S^1)$. Now by Lemma 3.9, $\mathrm{Homeo}_0(S^1)$ has uniformly bounded commutator length. Hence, we conclude by Proposition 1.18 that $\varphi - \psi$ is trivial. □

If we consider an element f in $\mathrm{Homeo}_+(S^1)$, the translation numbers of its lifts in $\widetilde{\mathrm{Homeo}}_+(S^1)$ differ by integers so that the element $\rho(f) = \tau(\tilde{f}) \mod \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ is well-defined.

Definition 3.11 (Rotation Number)**.** We call $\rho(f)$ the *rotation number* of f.

Example 3.12. If f in $\mathrm{Homeo}_+(S^1)$ has a fixed point x and $\tilde{f}$ is a lift of f, then $T(x) = x + k$ for some $k \in \mathbb{Z}$. Hence $T(\tilde{f}) = \tilde{f}(x) - x = k$ (see Remark 3.6). So f has rotation number $\rho(f) = 0$.

Talk 4: Quasimorphisms on the automorphism group of the disc

Jiahui, Konstantin

A. Ruelle's invariant on $\mathrm{Diff}_0(D^2, \partial D^2, area)$

Definition 4.1. $\mathrm{Diff}_0(D^2, \partial D^2, area)$ denotes the group of C^∞-diffeomorphisms on the unit disk $D^2 \subseteq \mathbb{R}^2$, that preserve the area form $area = dx \wedge dy$, and are identity in a neighbourhood of the boundary ∂D^2.

Our first goal is to construct a non-trivial quasimorphism on $\mathrm{Diff}_0(D^2, \partial D^2, area)$. Recall that if a non-trivial quasimorphism exists, then the commutator length is unbounded. The area form can in fact be any volume form by a result of Moser [Mos65]. A fact we assume about this group is that it is contractible [Sma59], which implies any $g \in \mathrm{Diff}_0(D^2, \partial D^2, area)$ is isotopic to id, which justifies our notation Diff_0 instead of Diff. This space is often studied before looking at diffeomorphisms on the sphere S^2.

We shall construct *Ruelle's invariant* $r : \mathrm{Diff}_0(D^2, \partial D^2, area) \to \mathbb{R}$, and show that it is a non-trivial quasimorphism. Let $g \in \mathrm{Diff}_0(D^2, \partial D^2, area)$, together with an isotopy $(g_t)_{t\in[0,1]}$ between $g_0 = id, g_1 = g$. Consider the differential of g_t at $x \in D^2$, $dg_t(x) : T_xD^2 \to T_{g_t(x)}D^2$. With the trivial coordinate chart of the disk we have a natural trivialization of the tangent bundle, so $dg_t(x)$ can be viewed as a linear map on $\mathbb{R}^2$. Since g_t are all area preserving, we in fact have $dg_t(x) \in \mathrm{SL}(2, \mathbb{R})$.

Let $u \in \mathbb{R}^2 - \{0\}$, then $dg_t(x)u$ is a non-zero vector because $dg_t(x) \in \mathrm{SL}(2, \mathbb{R})$, and thus varying t gives us a curve in $\mathbb{R}^2 - \{0\}$. Let $(A_{g_t}(x, u))_{t\in[0,1]}$ be a curve in $\mathbb{R}$ that represents the angle of $(dg_t(x)u)_{t\in[0,1]}$ with $A_{g_0}(x, u) = 0$, which exists by lifting property. Write $Ang_g(x) := A_{g_1}(x, \partial_1)$. Note that if we pick two different paths from id to g, then they are homotpic by contractibility, resulting a homotopy between the curves of angles. Since homotopies fix endpoints, $Ang_g(x)$ stays the same. Therefore $Ang_g(x)$ is independent of the path (g_t), and our notation is justified.

Next we shall show $Ang_g(x)$ is "almost" a quasimorphism. Let $g, h \in \mathrm{Diff}_0(D^2, \partial D^2, area)$ with isotopies $(g_t), (h_t)$. Consider the path from id to gh by first going from id to h via (h_t), then go from h to gh via (g_th). We have

$$Ang_{gh}(x) = A_{gh}(x, \partial_1) = A_h(x, \partial_1) + A_g(h(x), dh(x)\partial_1)$$

$$|Ang_{gh}(x) - Ang_h(x) - Ang_g(h(x))| = |A_g(h(x), dh(x)\partial_1) - A_g(h(x), \partial_1)| < \pi$$

where the inequality follows from the following lemma applied to $(dg_t(h(x)))_{t\in[0,1]}$.

Lemma 4.2. *Let $(U_t)_{t\in[0,1]}$ be a curve in $\mathrm{GL}(2, \mathbb{R})$. If $u, v \in \mathbb{R}^2 - \{0\}$, then angle of variation of $(U_t(u))$ and $(U_t(v))$ do not differ by more than π.*

Proof. If $u, v \in \mathbb{R}^2 - \{0\}$ are scalar multiples of each other, then the angle variation of the curves $(U_t(u)), (U_t(v))$ are the same. On the other hand, if u, v are linearly independent, then $U_t(u), U_t(v)$ are linearly independent at each t. This means the difference between angle variations of the two curves $(U_t(u)), (U_t(v))$ can not exceed π, otherwise by intermediate value theorem $U_t(u)$ and $U_t(v)$ must lie on the same line at some time t. □

To get the desiered quasimorphism, we need to eliminate the variable x in $Ang_g(x)$. This is easily done by taking

$$r(g) = \int_{D^2} Ang_g(x)d\ area(x)$$

$$\begin{aligned}
|r(gh) - r(h) - r(g)| &= \left| \int_{D^2} (Ang_{gh}(x) - Ang_h(x) - Ang_g(x))\, darea(x) \right| \\
&= \left| \int_{D^2} (Ang_{gh}(x) - Ang_h(x) - Ang_g(h(x)))\, darea(x) \right|, \text{ since } h \text{ preserves area} \\
&\leq \int_{D^2} |Ang_{gh}(x) - Ang_h(x) - Ang_g(h(x))| darea(x) \\
&\leq \pi \int_{D^2} darea = \pi^2
\end{aligned} \tag{8}$$

Homogenize r and get *Ruelle's homogeneous quasimorphism*

$$\mathfrak{Ruelle}(g) = \lim_{p\to\infty} \frac{1}{p} r(g^p)$$

It remains to show that this is non-trivial, which is expected as r measures "how much g rotates the disc on average". Let $\omega : [0,1] \to \mathbb{R}$ be a smooth map that is zero on some neighbourhoods of 0 and 1. Define area-preserving diffeomorphism F_ω on the disc (in polar coordinates) by

$$F_\omega(R,\theta) = (R, \theta + \omega(R))$$

Then we see

$$r(F_\omega) = \int_0^1 \int_0^{2\pi} Ang_{F_\omega}(R,\theta) R d\theta dR = 2\pi \int_0^1 \omega(R) R dR$$

Thus $\mathfrak{Ruelle}$ is non-trivial for r is unbounded.

B. The pure braid group $P_n(D^2)$

Fix n distinct points $x_1^0, ..., x_n^0$ in D^2. Let $X_n(D^2)$ be the space of n-tuples of distinct points in D^2. The fundamental group of $X_n(D^2)$ based at $(x_1^0, ..., x_n^0)$ is the *pure braid group*, denoted $P_n(D^2)$. We will use it to create an infinite set of linearly independent homogeneous quasimorphisms.

Elements of $P_n(D^2)$ are loops based at $(x_1^0, ..., x_n^0)$ with distinct coordinates, which correspond to n disjoint loops in the solid torus $D^2 \times S^1$. Thus we can associate a collection of n oriented circles in $\mathbb{R}^3$, known as a link in knot theory, to each pure braid. Such a link associated to pure braid γ, denoted $\hat{\gamma}$, is called a *closed pure braid*.

C. More Quasimorphisms on the Disc

Theorem 4.3. *For every closed oriented surface Σ, there exist homogeneous quasimorphisms $\Phi : \mathrm{Diff}_0^\infty(\Sigma, area) \to \mathbb{R}$ which are non-trivial, even when restricted to the kernel of Calabi's homomorphism (for $\Sigma \neq \mathbb{S}^2$). Moreover, the vector space of these homogeneous quasimorphisms is infinite dimensional.*

The goal of this section is to construct quasimorphisms on $\mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$. We will not only construct infinitely many of them but also prove that they are linearly independent.

C.1. The signature of a pure braid

In order to construct these quasimorphisms we will first describe a very important invariant of an oriented torus link, its *signature*.

Definition 4.4. At a crossing point c between two braids of an oriented regular diagram, there are two possible configurations. In case (a) we assign $s(c) = +1$ to the crossing point, while in case (b) we assign $s(c) = -1$. We call the crossing point in case (a) positive and in case (b) negative.

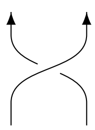


FIGURE 1. (a) $s(c) = +1$

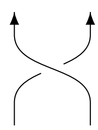


FIGURE 2. (b) $s(c) = -1$

Suppose now that we have an oriented regular diagram D of a torus link with two components K_1, K_2. Moreover, suppose that the crossing points of the projections of K_1 and K_2, so where the projections intersect, are $c_1, \ldots, c_n$. Then we call

$$lk(K_1, K_2) := \frac{1}{2}(s(c_1) + \cdots + s(c_n))$$

the *linking number* of K_1 and K_2.

Remark 4.5. The linking number is always an integer.

Let $\hat{\gamma} \subset \mathbb{R}^3$ be a torus link. We can choose a *Seifert surface* $S_{\hat{\gamma}}$ for $\hat{\gamma}$. A *Seifert surface* for $\hat{\gamma}$ is an oriented surface embedded in $\mathbb{R}^3$ whose oriented boundary is $\hat{\gamma}$. A theorem (1930) by Frankl and Pontryagin assures the existence of a Seifert surface for a torus link. We will now equip the first homology group $H_1(S_{\hat{\gamma}}, \mathbb{Z})$ with a bilinear form $B_{\hat{\gamma}}$ in the following way: For two closed oriented curves x and y on the surface $S_{\hat{\gamma}}$, $B_{\hat{\gamma}}(x, y)$ is just the linking number between x and $\tilde{y}$, where $\tilde{y}$ is the curve obtained by pushing the curve y a little bit away from the surface in the positive normal direction to $S_{\hat{\gamma}}$ (to ensure that the curves x and y do not intersect). It is easy to see that $B_{\hat{\gamma}}(x, y)$ only depends on the homology classes of the curves x and y, hence $B_{\hat{\gamma}}$ is a well-defined bilinear map on $H_1(S_{\hat{\gamma}}, \mathbb{Z})$. By the Universal Coefficients theorem, tensoring $H_1(S_{\hat{\gamma}}, \mathbb{Z})$ with $\mathbb{R}$ gives the vector space $H_1(S_{\hat{\gamma}}, \mathbb{R}) \approx H_1(S_{\hat{\gamma}}, \mathbb{Z}) \otimes \mathbb{R}$ (since $\mathbb{R}$ is a field) and we can turn $B_{\hat{\gamma}}$ into a symmetric bilinear form $\tilde{B}_{\hat{\gamma}}$ by $\tilde{B}_{\hat{\gamma}}(x, y) := B_{\hat{\gamma}}(x, y) + B_{\hat{\gamma}}(y, x)$ on this vector space.

Definition 4.6. For a symmetric bilinear form B with non-zero eigenvalues $\lambda_1 \geq \dots \lambda_p > 0 > \mu_1 \geq \dots \geq \mu_q$ we define the signature of B as $sign(B) := p - q$.

One can actually show that the signature of our symmetric bilinear form $\tilde{B}_{\hat{\gamma}}$ defined above does not depend on the Seifert surface we choose. Therefore for every pure braid γ and the associated closed torus link $\hat{\gamma}$ we can define $sign(\hat{\gamma}) := sign(\tilde{B}_{\hat{\gamma}}) \in \mathbb{Z}$.

Proposition 4.7. *The mapping* $sign : P_n(D^2) \to \mathbb{Z}, \gamma \mapsto sign(\hat{\gamma})$ *is a quasimorphism.*

Proof. Let $\alpha \in P_n(D^2)$ be a pure braid. We can find a Seifert surface $S_{\hat{\alpha}}$ so that outside of a cylinder in $\mathbb{R}^3$ the surface consists of n disjoint surfaces $D_{\alpha,x_1^0}, \dots D_{\alpha,x_n^0}$ all homeomorphic to disks. Now take another pure braid $\beta \in P_n(D^2)$ and its Seifert surface $S_{\hat{\beta}}$. An easy way to construct a Seifert surface $S_{\widehat{\alpha.\beta}}$for the pure braid $\alpha.\beta$ is by glueing the surfaces D_{α,x_i^0} and D_{β,x_i^0} along an interval along their boundary. Since $\alpha.\beta \in P_n(D^2)$ $S_{\widehat{\alpha.\beta}}$ has n boundary components and by the Mayer-Vietoris exact sequence applied to the two subsets A, B (essentially corresponding to $S_{\hat{\alpha}}, S_{\hat{\beta}} \subset S_{\widehat{\alpha.\beta}}$ respectively) whose intersection are n disjoint neighborhoods of all the intervals along which we glued the Seifert surfaces of α and β in the first place, we know that the first homology group of $S_{\widehat{\alpha.\beta}}$ contains a copy of the sum of the first homology groups of $S_{\hat{\alpha}}$ and $S_{\hat{\beta}}$ with codimension, at most, $n - 1$.
Now we can conclude the proof by observing that if we restrict a symmetric bilinear form to a subspace of codimension q, the signature of the restriction can change by, at most, q. Therefore $|sign(\alpha.\beta) - sign(\alpha) - sign(\beta)| \leq n - 1$. ☐

C.2. Constructing quasimorphisms on the disk

Now we are ready to actually construct many linearly independent homogeneous quasimorphisms on the group $\mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$. We do this by using the motion of n points, where $n \in \mathbb{N}$ is fixed, but arbitrary.

Let us therefore first fix n distinct points $(x_1, \dots, x_n)$ in the disk D^2, an element $g \in \mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$, as well as an isotopy $g_t, t \in [0, 1]$ from $g_0 = id$ to $g_1 = g$. We get a pure braid γ in $P_n(D^2)$ by the concatenation of the following three parts:

- $t \in [0, \frac{1}{3}] \mapsto ((1 - 3t)x_i^0 + 3tx_i)_{i=1,\dots,n} \in X_n(D^2)$
- $t \in [\frac{1}{3}, \frac{2}{3}] \mapsto (g_{3t-1}(x_i))_{i=1,\dots,n} \in X_n(D^2)$
- $t \in [\frac{2}{3}, 1] \mapsto ((3 - 3t)g(x_i) + (3t - 2)x_i^0)_{i=1,\dots,n} \in X_n(D^2)$

For almost all choices of points $(x_1, \dots, x_n)$ this gives indeed a loop in $X_n(D^2)$ and therefore defines a pure braid γ. Moreover, since $\mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$ is contractible the constructed braid does not depend on the isotopy g_t, and we denote this braid by $\gamma(g; x_1, \dots, x_n)$ and the associated closed torus link again by $\gamma(g; \widehat{x_1, \dots}, x_n)$.

We will now describe a linear map $\mathscr{G} : \mathcal{Q}(P_\bullet(D^2)) \to \mathcal{Q}(\mathrm{Diff}_0^\infty(D^2, \partial D^2, area))$ which associates to every quasimorphism on the pure braid group, on arbitrary many strands, of the disk a quasimorphism on $\mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$.
So suppose we are given a quasimorphism $\mu : P_n(D^2) \to \mathbb{R}$ with defect D_μ. It is easy to see, that for almost every points $(x_1, \dots, x_n)$ and diffeomorphisms $g, h \in \mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$ we have

$$\gamma(gh; x_1, \ldots, x_n) = \gamma(h; x_1, \ldots, x_n).\gamma(g; h(x_1), \ldots, h(x_n))$$

and hence

$$|\mu(\gamma(gh; x_1, \ldots, x_n)) - \mu(\gamma(h; x_1, \ldots, x_n)) - \mu(\gamma(g; h(x_1), \ldots, h(x_n))| \leq D_\mu.$$

Again, in order to get rid of the dependency of the exact choice of points $(x_1, \ldots, x_n)$, we define $\mathscr{G}(\mu)$ on every diffeomorphism $g \in \mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$ as follows

$$\mathscr{G}(\mu)(g) := \int \cdots \int \mu(\gamma(g; x_1, \ldots, x_n)) d\ area(x_1) \ldots d\ area(x_n) \in \mathbb{R}$$

and we remark that the set of points for which the integrand is not defined is a null-set. As all the integrals are finite and μ is a quasimorphism on $P_n(D^2)$, $\mathscr{G}(\mu)$ is a quasimorphism on the disk. (The calculation is similar to (8) and uses that g preserves area. Furthermore, note the dependency on n, the number of points we choose.

Definition 4.8. For a fixed $n \in \mathbb{N}$ and a given quasimorphism $\mu : P_n(D^2) \to \mathbb{R}$ we define the homogeneous quasimorphism $\mathscr{G}_h(\mu)$ as the homogenization of $\mathscr{G}(\mu)$, i.e.

$$\mathscr{G}_h(\mu)(g) := \lim_{p \to +\infty} \frac{1}{p} \mathscr{G}(\mu)(g^p)$$

for every diffeomorphism $g \in \mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$.

Example 4.9. Let us look at the example of μ being $sign : P_n(D^2) \to \mathbb{R}$, the quasimorphism we constructed in section C.1. Define $\mathfrak{Sign}_{n,D^2} := \mathscr{G}_h(sign)$.
We then have

$$\begin{aligned}
\mathfrak{Sign}_{n,D^2}(g) &= \mathscr{G}_h(sign)(g) = \lim_{p \to +\infty} \frac{1}{p} \mathscr{G}(sign)(g^p) \\
&= \lim_{p \to +\infty} \frac{1}{p} \int \cdots \int sign(\gamma(g^p; x_1, \ldots, x_n)) d\ area(x_1) \ldots d\ area(x_n) \\
&= \int \cdots \int \lim_{p \to +\infty} \frac{1}{p} sign(\gamma(g^p; x_1, \ldots, x_n)) d\ area(x_1) \ldots d\ area(x_n) \\
&= \mathscr{G}(Sign)(g)
\end{aligned}$$

where $Sign$ denotes the homogenization of $sign$.
In particular $\mathscr{G}$ commutes with taking the homogenization.

Thus we have constructed an infinite family of quasimorphisms $(\mathfrak{Sign}_{n,D^2})_{n \in \mathbb{N}}$ on $\mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$, which are supposably all different from each other. Indeed, in the next subsection we will show that the are even linearly independent, and hence also non-trivial.

C.3. Linear independence of the constructed quasimorphisms

Proposition 4.10. *All homogeneous quasimorphisms $\mathfrak{Sign}_{n,D^2}$, for $n \in \mathbb{N}$, are linearly independent.*

In order to show the linear independence we will evaluate our quasimorphisms on the family of diffeomorphisms F_ω we already considered in section A.

Now let $x_1, \ldots, x_n$ be n distinct points in the disc, such that $|x_1| < |x_2| <, \ldots, < |x_n|$. For $i \in 2, \ldots, n$ denote by $\eta_{i,n}$ the pure braid in which the loop starting in x_i goes once around $x_1, \ldots, x_{i-1}$ in positive direction (see Figure 3). Observe that the $\eta_{i,n}$'s are commuting pure braids, because if $i < j$ the loop starting in x_j "fully encircles" the loop starting in x_i, no matter which loop we follow first. One can moreover show that the following holds:

$$Sign(\eta_{i,n}) = \begin{cases} 1 - i, & \text{if } i \text{ is odd} \\ -i, & \text{if } i \text{ is even} \end{cases}$$

To state the subsequent lemma, we need two more notations. First of all, let us denote with $a(r)$ the spherical normalized are of the disc D_r of radius r centered at the origin

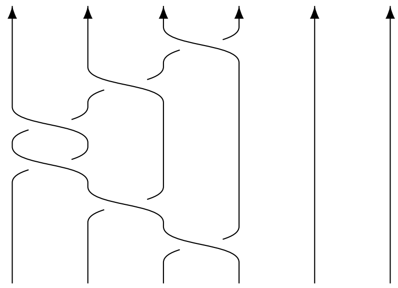


FIGURE 3. $\eta_{4,6}$ with $lk(\eta_{4,6}) = -4$

in $\mathbb{C}$ (i.e. if we identify the 2-sphere together with the usual (normalized) area form with $\overline{\mathbb{C}}$, the disc D_r can be considered as a disc on the 2-sphere and we choose the area form $a(r)$ to be the such that the pullback of D_r onto the 2-sphere has exactly total mass $a_(r)$. In other words, we push-forward the usual measure on the 2-sphere, for example with the standard stereographic projection onto $\overline{\mathbb{C}}$. An easy computation shows $a(r) = \frac{r^2}{1+r^2}$). If we now define the function u by the relation $a = \frac{1-u}{2}$ and set $\overline{\omega}(u(r)) := \omega(r)$ the following lemma reads as follows:

Lemma 4.11. $\mathfrak{Sign}_{n,D^2}(F_\omega) = \frac{n}{4}\int_{-1}^{+1}(u^{n-1} + (n-1)u - n)\bar{\omega}(u)\ du.$

Proof. There exists a constant $M(n) > 0$ (depending on n) such that for all $p > 0$ and $x_1 \dots, x_n$ with $|x_1|, \dots, |x_n|$, the pure braid $\gamma(F^p_\omega; x_1, \dots, x_n)$ can be decomposed as follows:

$$\gamma(F^p_\omega; x_1, \dots, x_n) = \gamma_1.\eta_{2,n}^{[\omega(|x_2|)p]} \dots \eta_{n,n}^{[\omega(|x_n|)p]}.\gamma_2$$

where $|Sign(\gamma_1)| < M(n)$, $|Sign(\gamma_1)| < M(n)$ and $[-]$ denotes the integer part. Recall that homogeneous quasimorphisms restrict to homomorphisms on Abelian subgroups and since all $\eta_{i,n}$ commute, we have

$$|Sign(\gamma(F^p_\omega; x_1, \dots, x_n)) - \sum_{i=2}^{n} Sign(\eta_{i,n})[\omega(|x_i|)p]| \le 2M(n) + 2D_{Sign} \le 2M(n) + 2(n-1).$$

Therefore we get

$$\mathfrak{Sign}_{n,D^2}(F_\omega) = (n!)\int \cdots \int_{|x_1|<\cdots<|x_n|} \sum_{i=2}^{n} Sign(\eta_{i,n})\omega(|x_i|)\ d\ area(x_1) \dots d\ area(x_n)$$

and, hence

$$\mathfrak{Sign}_{n,D^2}(F_\omega) = \int_0^1 \sum_{i=2}^{n} Sign(\eta_{i,n}) i \binom{n}{i} (a(r))^{i-1}(1-a(r))^{n-i}\omega(r)\ d\ a(r)$$

Now by a change of variables $a = (1-u)/2$, using the values $Sign(\eta_{i,n}) = 1-i$ for odd i and $Sign(\eta_{i,n}) = -i$ for even i, as well as a tedious calculation gives the formula stated in the lemma.

□

Note that since we have a lot of freedom to choose the function ω, $\mathfrak{Sign}_{n,D^2}$ is definitely non-trivial. We can finally prove proposition 4.10 from above and hence finish the goal of this section

Proof of proposition 4.10. It is sufficient to note that for $\mathfrak{Sign}_{n,D^2}(F_\omega)$ the polynomial in u, in the previous lemma, has a non-zero term of degree n. Hence there cannot be a vanishing linear combination of this family of quasimorphisms. □

Talk 5: Quasimorphisms on the automorphism groups of closed oriented surfaces

Lukas, Adrian

A. Main Result

The main result we are going to prove generalizes last weeks result on the disc to closed surfaces of genus $g \geq 1$. We follow closely [GG04, sections 2.2 and 3].

Theorem 5.1 ([GG04])**.** *For every closed oriented surface Σ of genus ≥ 1, there exist homogeneous quasimorphisms $\Phi : \mathrm{Diff}_0^\infty(\Sigma, area) \to \mathbb{R}$ which are non-trivial.*

Remark 5.2.
- The same is true for S^2, but we won't prove it.
- There are actually infinitely many linearly independent homogeneous quasimorphisms for $\Sigma \neq S^2$.
- In fact, there is a homomorphism

  $$\mathrm{flux}\colon \mathrm{Diff}_0^\infty(\Sigma, area) \to H_1(\Sigma, \mathbb{R})$$

  whose kernel is simple. The quasimorphisms we construct will be non-trivial on this kernel.

For this we treat the cases of genus 1 (the torus) and higher genus separately.

B. Torus

In this section we will view the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the area form $area = dx \wedge dy$ and so we can trivialize the tangent space as $\mathbb{R}^2$ similarly to how it was done for the disc. By a result from Moser [Mos65] the choice of area form does not matter.
So we now construct Ruelle's quasimorphism on $\mathrm{Diff}_0(\mathbb{T}^2, area)$ in a manner heavily mirroring the construction for the disc. We start with an element $g \in \mathrm{Diff}_0(\mathbb{T}^2, area)$ and an isotopy $(g_t)_{t\in[0,1]}$ in $\mathrm{Diff}_0(\mathbb{T}^2, area)$ from $g_0 = id$ to $g_1 = g$ and take a point $x \in \mathbb{T}^2$.
Given our trivialization of $T\mathbb{T}^2 = \mathbb{R}^2$ and the fact that the g_t are all area preserving we may consider $dg_t(x) \in \mathrm{SL}(2, \mathbb{R})$ and can thus observe the path $\frac{dg_t(x)\partial_x}{||dg_t(x)\partial_x||}$ in S^1. We can now lift this path to the universal cover of S^1, namely $\mathbb{R}$, such that $dg_0(x)\partial_x = (1, 0)$ is lifted to 0 and denote by $Ang_g(x)$ the endpoint at $t = 1$ of this path. To show that it is independent of the choice of isotopy we first need a lemma.

Lemma 5.3. *View $\mathbb{T}^2 \subseteq \mathrm{Diff}_0(\mathbb{T}^2, area)$ as the subgroup of translations. This inclusion is a homotopy equivalence.*

A proof can be found in [EE67] and will not be given here. Now we may consider two isotopies $(g_t)_{t\in[0,1]}$ and $(g_t')_{t\in[0,1]}$ from the identity to g and form a loop $h_t \in \mathrm{Diff}_0(\mathbb{T}^2, area)$ first going along g_t and in reverse along g_t'. On the one hand we obtain $Ang_h(x) = Ang_g(x) - Ang_{g'}(x)$. On the other hand a loop of translations would result in a constant curve in S^2 as the differentials are constantly the identity, so since h is homotopic to such a loop (Lemma 5.3) we have $Ang_h(x) = 0$. So $Ang_g(x) = Ang_{g'}(x)$ and it is independent of the isotopy.
As with the disc we now get:

$$|Ang_{gh}(x) - Ang_h(x) - Ang_g(h(x))| < \pi$$

Which means $r(g) = \int_{\mathbb{T}^2} Ang_g(x) darea(x)$ defines a quasimorohism with $|r(gh) - r(h) - r(g)| \leq \pi$, following the same calculations and we can homogenize it to obtain:

$$\mathfrak{Ruelle}(g) = \lim_{p\to\infty} \frac{1}{p} r(g^p)$$

Our next goal is to show that this quasimorphism is non-trivial, for which we will use the fact that our construction is nearly identical to that of $\mathfrak{Ruelle}$ on the disc.

We choose a small embedded disc D^2 in the torus (e.g. $\bar{B}_{\frac{1}{4}}(\frac{1}{2}, \frac{1}{2})$) and note that since any element $g \in \mathrm{Diff}_0(D^2, \partial D^2, area)$ is the identity near the boundary of the disc, g can be smoothly extended to the whole of the torus via the identity. We now see that the trivializations of the tangent bundles are the same, so since the rest of the construction is the same, we have that $\mathrm{Diff}_0(D^2, \partial D^2, area) \subseteq \mathrm{Diff}_0(\mathbb{T}^2, area)$ is a subgroup on which $\mathfrak{Ruelle}$ is non-trivial.

C. Construction of higher genus surfaces

Before moving on to the surfaces of higher genus we recall a useful construction, which will allow us to use a similar strategy as in the previous cases of the disc and the torus, since we cannot trivialize the tangent bundle of higher genus surfaces.

A common construction of a genus g surface is by a side pairing of a $4g$-gon, where each set of 4 sequential sides add one to the genus (for a good visualization of this, see `https://math.stackexchange.com/a/3427810`).

Another common construction is the tiling of the hyperbolic plane by any regular polygons (for the example of the 8-gon, with the pairing from before, see figures 1 and 2 in [Gol12].

The side pairings of such a tiling with the $4g$-gon give us a group of isometries acting on the hyperbolic plane and thus a cover from it to the surface of genus g.

The Poincaré Polygon Theorem gives us that this is a Fuchsian group and since the genus is at least 2 we have that the cover is indeed a metric cover and we now have a structure on our genus g surface with curvature of -1.

D. Surfaces of higher genus

The next goal is to construct a quasimorphism on the automorphism group $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ of a closed genus $g \geq 2$ surface Σ. As always $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ denotes the group of C^∞-diffeomorphisms of Σ onto itself that preserve a specific area form area on Σ and are in the path-component of the identity. The idea for the construction of a quasimorphism on this group is again to generalize the Ruelle quasimorphism. However we will see that the situation in this case cannot be reduced to a computation in $\mathbb{R}^m$, so that we have to take a slightly different approach. First we introduce some necessary facts:

Lemma 5.4. *The group* $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ *is contractible for any closed oriented surface* Σ *of genus* $g \geq 2$.

A proof of this lemma can be found in [EE67] but will not be given here. Another fact that we will need to use is that any surface[1] Σ of genus $g \geq 2$ can be given a hyperbolic metric of constant curvature -1. Even further (a consequence of the Killing–Hopf theorem):

Lemma 5.5. *Let* Σ *be a surface of genus* $g \geq 2$ *with a suitable metric of curvature* -1. *The universal cover* $\tilde{\Sigma}$ *seen as a Riemannian cover of* Σ *is the hyperbolic disk* $\mathbb{D}^2$ *with its usual hyperbolic metric.*

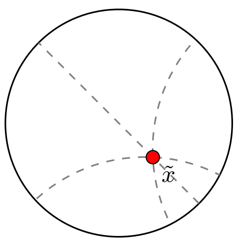


FIGURE 4. Three geodesics through a point $\tilde{x} \in \mathbb{D}^2$ in the hyperbolic disk.

Lemma 5.5 hints at a way to reduce our situation to computations on the disk, where we have already defined a quasimorphism last week, cf. talk 4. In the following let Σ denote a surface of genus $g \geq 2$ with a suitable hyperbolic metric. The idea is the following: Given $g \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ we choose an isotopy $g_t : [0,1] \times \Sigma \to \Sigma$ with $g_0 = \mathrm{id}$ and $g_1 = g$. We now lift to an isotopy of diffeomorphisms of the universal (hyperbolic) cover $\tilde{\Sigma} = \mathbb{D}^2$: Namely $\tilde{g} : [0,1] \times \mathbb{D}^2 \to \mathbb{D}^2$, were we use the notation $\tilde{g}_t(\tilde{x})$ for $t \in [0,1]$ and $\tilde{x} \in \mathbb{D}^2$. Now that we have found a way to work on the disk again, we want to count rotations as before. However we cannot use exactly the same method as for the usual disk: this would disregard the metric structure. To make use of the metric, recall how geodesics look on the hyperbolic disk: they either are straight lines (in the usual sense) that connect opposite points on the circle bounding the hyperbolic disk or they are themselves circle segments that meet the boundary circle perpendicularly. Figure 4 shows the situation. Since we work with a Riemannian cover these geodesics also behave nicely with respect

[1] For the rest of this talk *surface* shall mean *closed, oriented surface.*

to the cover. As they all meet the boundary circle $\partial\mathbb{D}^2$ these geodesics give us a way to simplify our setting further. Given any point and a tangent vector we get a unique geodesic and thus a point on the boundary circle. This is formalized in the following definition:

Definition 5.6. We define the following map

$$\pi : T\mathbb{D}^2 \setminus 0 \to \partial\mathbb{D}^2 \cong S^1$$
$$(\tilde{x}, \tilde{v}) \mapsto \lim_{t\to\infty} \exp_{\tilde{x}}(t\tilde{v}).$$

This definition gives us the tools to study (a neighborhood of) a point $x \in \mathbb{D}^2$ in the following way: the differential of the diffeomorphism $\tilde{g}_t : \mathbb{D}^2 \to \mathbb{D}^2$ at $\tilde{x}$ is an isomorphism $d\tilde{g}_t(\tilde{x}) : T_{\tilde{x}}\mathbb{D}^2 \to T_{\tilde{g}_t(\tilde{x})}\mathbb{D}^2$. So if we choose any non-zero tangent vector $\tilde{v} \in T_{\tilde{x}}\mathbb{D}^2$, we can track how $\pi(\tilde{g}_t)(\tilde{x}), d\tilde{g}_t(\tilde{x})(\tilde{v}))$ moves as we follow the isotopy $\tilde{g}_t$ from $t = 0$ to $t = 1$. This motivates the following definition:

Definition 5.7. Let $\tilde{g}_t$ be as above and $(\tilde{x}, \tilde{v}) \in T\mathbb{D}^2$, then we define the curve $\gamma_{(\tilde{x},\tilde{v})} : [0, 1] \to S^1$ as follows: $t \mapsto \pi(\tilde{g}_t(\tilde{x}), d\tilde{g}_t(\tilde{x})(\tilde{v}))$.

Since we are now looking at a curve on $\partial\mathbb{D}^2 \cong S^1$ we can count the number of *full turns* this curve makes around the circle by lifting it to the universal cover $\mathbb{R}$. This motivates the following definition which will later take the role of the angle in the definition of the Ruelle invariant for the genus g surface.

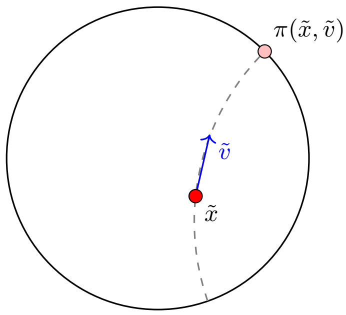


FIGURE 5. A pair $(\tilde{x}, \tilde{v}) \in T\mathbb{D}^2$ and the point $\pi(\tilde{x}, \tilde{v})$ on the boundary $\partial\mathbb{D}^2$.

Definition 5.8. Let $g \in \mathrm{Diff}_0(\Sigma, \text{area})$ and $(\tilde{x}, \tilde{v}) \in T\mathbb{D}^2 \setminus 0$ and let $\gamma_{(\tilde{x},\tilde{v})}$ be defined as above for g. Further assume that $\tilde{\gamma}_{(\tilde{x},\tilde{v})} : [0, 1] \to \mathbb{R}$ is a lift of $\gamma_{(\tilde{x},\tilde{v})}$, then we set

$$r(g, \tilde{x}, \tilde{v}) = \lfloor \tilde{\gamma}_{(\tilde{x},\tilde{v})}(1) - \tilde{\gamma}_{(\tilde{x},\tilde{v})}(0) \rfloor.$$

Lemma 5.9. *The map r from definition 5.8 is well-defined, i.e. it does not depend on the choice of lift $\tilde{\gamma}$ nor on the choice of isotopy $\tilde{g}_t$.*

Proof. The independence with respect to the choice of lift $\tilde{\gamma}$ is clear, so we only need to show that the choice of isotopy $\tilde{g}_t$ does not matter. So let $g_t^{(1)}, g_t^{(2)}$ be two isotopies from the identity to $g \in \mathrm{Diff}_0(\Sigma, \text{area})$. Since the group is contractible there is an isotopy of isotopies

$$H : [0, 1]^2 \times \Sigma \to \Sigma$$

with $H(0, t) = g_t^{(1)}$ and $H(1, t) = g_t^{(2)}$. Then we can lift to the universal cover and obtain a map

$$\tilde{H} : [0, 1]^2 \times \mathbb{D}^2 \to \mathbb{D}^2.$$

We can now consider the "projections" to the boundary circle all at once by looking a the map

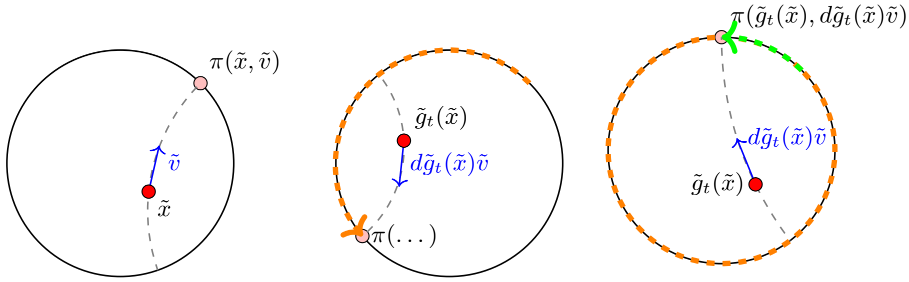
$$\Gamma : [0, 1]^2 \to S^1$$
$$s, t \mapsto \pi(H_{s,t}(\tilde{x}), dH_{s,t}(\tilde{x})(\tilde{v})).$$

Let $\tilde{\Gamma}$ be the lift of this map to the universal cover $\mathbb{R}$ of S^1. Now by construction $\tilde{\Gamma}(0, \cdot)$ is a valid choice for $\tilde{\gamma}^{(1)}$ and $\tilde{\Gamma}(1, \cdot)$ for $\tilde{\gamma}^{(2)}$. Since $H(\cdot, 0) = id$ and $H(\cdot, 1) = g$ we have that $\tilde{\Gamma}(\cdot, 0)$ and $\tilde{\Gamma}(\cdot, 1)$ are constant which finishes the proof. □

The first thing we can say about the behavior of this r is this easy lemma:

Lemma 5.10. *Let $g \in \mathrm{Diff}_0(\Sigma, \text{area}), \tilde{x} \in \mathbb{D}^2$ and $\tilde{v}_1, \tilde{v}_2 \in T_{\tilde{x}}\mathbb{D}^2 \setminus \{0\}$. Then*

$$|r(g, \tilde{x}, \tilde{v}_1) - r(g, \tilde{x}, \tilde{v}_2)| \leq 1.$$



(A) $t = 0$ (B) $t = \frac{1}{2}$ (C) $t = 1$

FIGURE 6. Visualization of computing $r(g, \tilde{x}, \tilde{v}) = 1$, the first full turn is marked in orange.

Proof. We show the lemma by contradiction. Assume that $|r(g, \tilde{x}, \tilde{v}_1) - r(g, \tilde{x}, \tilde{v}_2)| > 1$ for some $g \in \mathrm{Diff}_0(\Sigma, \mathrm{area}), \tilde{x} \in \mathbb{D}^2$ and non-zero $\tilde{v}_1, \tilde{v}_2 \in T_{\tilde{x}}\mathbb{D}^2$. Then there exists a $t \in [0, 1]$ such that $d\tilde{g}_t(\tilde{x})(\tilde{v}_1) = d\tilde{g}_t(\tilde{x})(\tilde{v}_2)$ by uniqueness of the geodesics given the initial conditions. Since $d\tilde{g}_t(\tilde{x})$ is an isomorphism this implies $\tilde{v}_1 = \tilde{v}_2$ which implies $|r(g, \tilde{x}, \tilde{v}_1) - r(g, \tilde{x}, \tilde{v}_2)| = 0$ and thus we arrive at a contradiction. $\square$

The natural next step is to show that this r behaves almost like a quasimorphism. This is achieved by the following lemma:

Lemma 5.11. *Let $g, h \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ and let $(\tilde{x}, \tilde{v}) \in T\mathbb{D}^2 \setminus 0$. Then*

$$\left| r(gh, \tilde{x}, \tilde{v}) - r(h, \tilde{x}, \tilde{v}) - r(g, \tilde{h}(\tilde{x}), d\tilde{h}(\tilde{x})(\tilde{v})) \right| \leq 1.$$

Proof. To show this property we will choose some especially nice isotopies, namely let g_t be an isotopy from the identity to g that remains the identity for all $t \in [0, \frac{1}{2}]$ and let h_t be an isotopy from the identity to h such that $h_t = h$ for all $t \in [\frac{1}{2}, 1]$. Now we clearly get an isotopy from the identity to gh by considering $t \mapsto g_t \circ h_t$. Using this we obtain the bound from the lemma as follows: We want to count the full turn of the curve

$$t \mapsto \pi(\tilde{g}_t\tilde{h}_t(\tilde{x}), d(\tilde{g}_t\tilde{h}_t(\tilde{x}))(\tilde{v})). \tag{9}$$

If we restrict t in (9) to $[0, \frac{1}{2}]$ this simplifies to $t \mapsto \pi(\tilde{h}_t(\tilde{x}), d\tilde{h}_t(\tilde{x})(\tilde{v}))$ and thus the number of full turns our curve above makes before time $t = \frac{1}{2}$ is exactly $r(h, \tilde{x}, \tilde{v})$. Now using the chain rule we obtain that for $t \in [\frac{1}{2}, 1]$ that (9) is the same as considering the curve $t \mapsto \pi(\tilde{g}_t(\tilde{x}), d\tilde{g}_t(\tilde{h}(\tilde{x}))(d\tilde{h}(\tilde{x})\tilde{v}))$. So the number of full turns that (9) makes after time $t = \frac{1}{2}$ is exactly $r(g, \tilde{h}(\tilde{x}), d\tilde{h}(\tilde{x})(\tilde{v}))$. This already implies our bound since that means that a difference between $r(gh, \tilde{x}, \tilde{v})$ and $r(h, \tilde{x}, \tilde{v}) + r(g, \tilde{h}(\tilde{x}), d\tilde{h}(\tilde{x})(\tilde{v}))$ can only come from unfinished turns summing up to a full term in the end. Since two unfinished turns cannot combine to more than a full turn this finishes the proof. $\square$

What is now left to do is to get rid of the dependence of r on $\tilde{v}$ and $\tilde{x}$. We first start by eliminating $\tilde{v}$:

Definition 5.12. Let $g \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ and $\tilde{x} \in \mathbb{D}^2$. Then we define

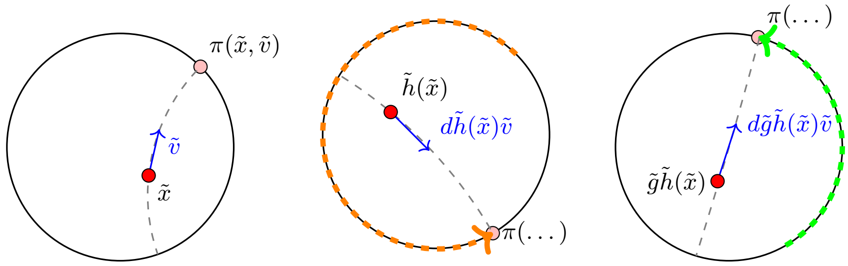
$$r(g, \tilde{x}) = \inf_{\tilde{v} \in T_{\tilde{x}}\mathbb{D}^2 \setminus \{0\}} r(g, \tilde{x}, \tilde{v}).$$

Given lemmas 5.11 and 5.10, we get the following corollary:

Corollary 5.13. *Let $g, h \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ and $\tilde{x} \in \mathbb{D}^2$. Then*

$$\left| r(gh, \tilde{x}) - r(h, \tilde{x}) - r(g, \tilde{h}(\tilde{x})) \right| \leq 4.$$

Now our work on the cover is done and we want to return to Σ. In order to do that, we note that if we have two points $\tilde{x}_1, \tilde{x}_2 \in \mathbb{D}^2$ that are lifts of the same point $x \in \Sigma$, there exists a deck transformation $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ s.t. $\tilde{x}_2 = \Phi(\tilde{x}_1)$. Obviously Φ descends to the identity on Σ. Thus for any $g \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ we obtain $r(g, \tilde{x}_2) = r(g, \Phi(\tilde{x}_1))$. Now



(A) $t = 0$ (B) $t = \frac{1}{2}$ (C) $t = 1$

FIGURE 7. Visualization of the proof of lemma 5.11. While $r(h, \tilde{x}, \tilde{v}) = r(g, \tilde{h}(\tilde{x}), d\tilde{h}(\tilde{x})(\tilde{v})) = 0$, the two unfinished turns sum up to a full turn thus giving $r(gh, \tilde{x}, \tilde{v}) = 1$.

let $\tilde{g}_t$ be any isotopy from the identity on $\mathbb{D}^2$ to a lift of g. Then $\Phi \circ \tilde{g}_t = \tilde{g}_t \circ \Phi$ and this isotopy is also an isotopy from the identity on $\mathbb{D}^2$ to a lift of g. Thus by definition we obtain $r(g, \tilde{x}_2) = r(g, \tilde{x}_1)$. And thus r is invariant under deck transformations and therefore descends to a map

$$r : \mathrm{Diff}_0(\Sigma, \mathrm{area}) \times \Sigma \to \mathbb{Z}.$$

Now we can already obtain a quasimorphism:

Definition 5.14. We define the map

$$\begin{aligned} r : \mathrm{Diff}_0(\Sigma, \mathrm{area}) &\to \mathbb{R} \\ g &\mapsto \int_\Sigma r(g, x) d\,\mathrm{area}(x). \end{aligned}$$

Proposition 5.15. *The map r defined in definition 5.14 is a quasimorphism.*

Proof. Let $g, h \in \mathrm{Diff}_0(\Sigma, \mathrm{area})$ be arbitrary. Using the fact that g is area-preserving we obtain

$$\begin{aligned} |r(gh) - r(h) - r(g)| &= \left| \int_\Sigma r(gh, x) - r(h, x) d\,\mathrm{area}(x) - \int_\Sigma r(g, x) d\,\mathrm{area}(x) \right| \\ &= \left| \int_\Sigma r(gh, x) - r(h, x) d\,\mathrm{area}(x) - \int_\Sigma r(g, h(x)) d\,\mathrm{area}(x) \right| \\ &\leq \int_\Sigma |r(gh, x) - r(h, x) - r(g, h(x))| d\,\mathrm{area}(x) \\ &\leq \int_\Sigma 4\,\mathrm{area} = 4\,\mathrm{area}(\Sigma) < \infty, \end{aligned}$$

which completes the proof. □

The Ruelle quasimorphism is now obtained by homogenization of r:

Definition 5.16. The homogeneous quasimorphism

$$\mathfrak{Ruelle}_\Sigma(g) := \lim_{p \to \infty} \frac{1}{p} r(g^p)$$

is called the *Ruelle quasimorphism*.

The following theorem justifies the effort we went through to define $\mathfrak{Ruelle}_\Sigma$:

Theorem 5.17. *Let $D \hookrightarrow \Sigma$ be a disk embedded with an area-preserving embedding. Then $\mathfrak{Ruelle}_\Sigma$ restricted to the subgroup $\mathrm{Diff}_0(D, \partial D, \mathrm{area}) < \mathrm{Diff}_0(\Sigma, \mathrm{area})$ agrees (up to area normalization) with the Ruelle quasimorphism on the disk as defined before and in particular $\mathfrak{Ruelle}_\Sigma$ is non-trivial.*

Proof. Let $g \in \mathrm{Diff}_0(D, \partial D, \mathrm{area}) < \mathrm{Diff}_0(\Sigma, \mathrm{area})$ be arbitrary. Note that we view g as an element of $\mathrm{Diff}_0(\Sigma, \mathrm{area})$ by extending it to $\Sigma \setminus D$ by the identity. This directly implies

$$r(g) = \int_\Sigma r(g, x) d\,\mathrm{area}(x) = \int_D r(g, x) d\,\mathrm{area}(x).$$

Using the fact that homogenous quasimorphisms with bounded difference are equal we only need to show that $Ang_g(x) - r(g, x)$ is bounded. But by construction this term is bounded and only depends on the embedding $D \hookrightarrow \Sigma$, thus proving the theorem. □

Remark 5.18. A little bit more could be said with more time: There exists a homomorphism $\mathrm{Diff}_0(\Sigma, \mathrm{area}) \to H^1(\Sigma; \mathbb{R})$ and the subgroup $\mathrm{Diff}_0(D, \partial D, \mathrm{area}) < \mathrm{Diff}_0(\Sigma, \mathrm{area})$ is contained in its kernel. A construction of this homomorphism can be found in section 2.3 of [GG04].

Talk 6: Introduction to symplectic geometry and the group of Hamiltonian diffeomorphisms

Huaitao, Patrik

Through out this talk, let's assume M to be a connected smooth manifold without boundary.

A. Symplectic manifolds and symplectomorphisms

Definition 6.1 (Symplectic manifold)**.** A *sympletic* manifold is a pair (M, ω), where M is a smooth manifold and ω is a closed non-degenerate 2-form on M.

Example 6.2.
- The basic example of a symplectic manifold is $M = \mathbb{R}^{2n}$ with the standard symplectic form
$$w_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$
where M is equipped with the standard linear coordinates $x_1, \dots, x_n, y_1, \dots, y_n$.
- Assume that M is an orientable manifold with $\dim(M) = 2$ and with volume form $\omega \in \Omega^2(M)$. We see that ω is closed and that it is non-degenerate since it is a volume form. We deduce that every orientable surface is a symplectic manifold, i.e. the sphere S^2 with the standard area form is a symplectic manifold. One can see more examples in [MS17].

Theorem 6.3 (Darboux, [Car13])**.** *Let (M, ω) be a symplectic manifold, not necessarily compact, and $p \in M$. Then there exist neighborhoods U of p in M and U' of the origin in $\mathbb{R}^{2n}$ and a diffeomorphism $\phi : U \to U'$ such that $\phi^* w = w_0$, where w_0 is the standard symplectic form on Euclidean space.*

A proof of this Theorem can be found in [MS17] but will not be given here.

Remark 6.4 (Symplectic manifolds are orientable, even-dimensional)**.** Darboux's Theorem says that every symplectic form ω on M is locally diffeomorphic to to the standard form ω_0 on $\mathbb{R}^{2n}$. Hence every symplectic manifold is even-dimensional. Since we know that M is even-dimensional, we can write $n = \frac{1}{2}\dim(M)$. Denote $\mathcal{L} = \frac{1}{n!}\omega^n$, called Liouville form. The Liouville form is a non-vanishing top-form and hence a volume form. We conclude that any symplectic manifold is oriented.

Definition 6.5 (Symplectomorphism)**.** A *symplectomorphism* of a symplectic manifold (M, ω) is a diffeomorphism $\phi \in \mathrm{Diff}(M)$ such that
$$\omega = \phi^* \omega.$$
So, any symplectomorphism ϕ preserves the symplectic form ω. We denote the group of symplectormorhpisms of (M, ω) by
$$\mathrm{Symp}(M, \omega) := \{\phi \in \mathrm{Diff}(M) | \phi^* \omega = \omega\}.$$
We use the symbol $\mathrm{Symp}_0(M, \omega)$ to denote the identity component of $\mathrm{Symp}(M, \omega)$, i.e. the group of symplectomorphisms that are isotopic to the identity through symplectomorphisms. We will usually omit ω and write $\mathrm{Symp}(M)$ for the group of symplectomorphisms.

B. Symplectic and Hamiltonian vector fields

Let X be a vector field defined over M. Let T be a tensor field defined over M. Let ρ_t be the local flow of X at time t. In other words, for every $p \in M, t \mapsto \rho_t(p)$ is the integral curve of X starting at p.

Definition 6.6. The *Lie derivative* of T with respect to X at a point $p \in M$ is given by
$$(\mathcal{L}_X T)_p = \frac{d}{dt}\Big|_{t=0} ((\rho_t)^* T)_p.$$
The Lie derivative has some nice properties, but the most useful property is:

Theorem 6.7 (Cartan's Formula)**.** *We have*

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega).$$

Remark 6.8. For a vector field X on M the map $\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$ which sends a p-form ω to the $(p-1)$-form $\iota_X w$ is defined by

$$(\iota_X\omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1})$$

for any vector fields $X_1, \dots, X_{p-1}$. $\iota_X\omega$ is called the interior product.

Definition 6.9 (Hamiltonian vector field)**.** Let (M, ω) be a symplectic manifold. If $H \in \mathrm{C}^\infty(M)$, then since $\tilde{\omega} : TM \to T^*M$ is an linear isomorphism (which follows from the fact that ω is non-degenerate), there is an unique vector field X_H on M such that

$$\iota_{X_H}\omega = dH.$$

X_H is called *the Hamiltonian vector field* with Hamiltonian function H.

Definition 6.10 (Symplectic vector field)**.** A vector field X on (M, ω) is called *symplectic* if $\iota_X\omega$ is closed.

Remark 6.11. Every Hamiltonian vector field is a symplectic vector field.

Proof. Let X_H be a Hamiltonian vector field. Then it immediately follows that X_H is symplectic since ι_{X_H} is exact and so in particular it is closed. □

Remark 6.12. There is an equivalent definition for symplectic vector fields, namely by defining symplectic vector fields to be a vector field X such that $\mathcal{L}_X\omega = 0$.

Proof. The statement in the remark is equivalent to show $d(\iota_X\omega) = 0 \Leftrightarrow \mathcal{L}_X\omega = 0$. Using Cartan's Formula we have

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) = d(\iota_X\omega)$$

The second equality follows from the fact that ω is closed ($d\omega = 0$). □

C. Symplectic and Hamiltonian isotopy

Definition 6.13. Let (M, ω) be a symplectic manifold. A symplectic isotopy of (M, ω) is a smooth map $[0, 1] \times M \to M, (t, q) \mapsto \psi_t(q)$ such that $\psi_t \in \mathrm{Symp}(M, \omega), \forall t \in [0, 1]$ and $\psi_0 = \mathrm{id}$.

Proposition 6.14. *If* $t \mapsto \psi_t \in \mathrm{Diff}(M)$ *is a smooth family of diffeomorphisms generated by a family of vector fields* $X_t \in \mathfrak{X}(M)$ *via* $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, $\psi_0 = \mathrm{id}$*, then* $\psi_t \in \mathrm{Symp}(M, \omega)$ *for every* t *if and only if* $X_t \in \mathfrak{X}(M, \omega)$ *for every* t*.*

Proof.

$$\begin{aligned}\frac{d}{dt}|_{t=t_0}\Psi^*\omega &= \lim_{t\to t_0}\frac{\Psi_t^*\omega - \Psi_{t_0}^*\omega}{t - t_0} = \Psi_{t_0}^*\lim_{t\to t_0}\frac{(\Psi_{t_0}^{-1})^*\Psi_t^*\omega - \omega}{t - t_0} \\ &= \Psi_{t_0}^*\lim_{t\to t_0}\frac{(\Psi_t \circ \Psi_{t_0}^{-1})^*\omega - \omega}{t - t_0} = \Psi_{t_0}^*\mathcal{L}_{X_{t_0}}\omega\end{aligned}$$

So Ψ_t is symplectic, $\forall t \Leftrightarrow \frac{d}{dt}\Psi^*\omega = 0, \forall t \Leftrightarrow \mathcal{L}_{X_t}\Psi_t = 0, \forall t \Leftrightarrow X_t$ is symplectic, $\forall t$. □

Remark 6.15 (symplectic isotopy)**.** On a closed manifold, there is a 1-1 correspondence between isotopies (smooth curves in $\mathrm{Diff}(M)$ passing through id at time 0) and time-dependent vector fields, namely $\{\Psi_t\} \subset \mathrm{Diff}(M), \Psi_0 = \mathrm{id} \leftrightarrow \{X_t\} \subset \mathfrak{X}(M)$. By definition, a symplectic isotopy is exactly such a curve with image in $\mathrm{Symp}(M, \omega)$. Then the proposition above tells us there is also a correspondence between symplectic isotopies and smooth families of symplectic vector fields, that is, $\{\Psi_t\} \subset \mathrm{Symp}(M, \omega) \leftrightarrow \{X_t\} \subset \mathfrak{X}(M, \omega)$.

This motivates the definition of Hamiltonian isotopy.

Definition 6.16 (Hamiltonian isotopy)**.** A symplectic isotopy $\{\Psi_t\}_{0\leq t\leq 1}$ is called a Hamiltonian isotopy if its corresponding vector fields $\{X_t\}$, such that $\frac{d}{dt}\Psi_t = X_t \circ \Psi_t$, is Hamiltonian for any t. In this case, there is a smooth function $H : [0, 1] \times M \to \mathbb{R}$ such that for each t, the function $H_t := H(t, \cdot)$ generates the vector field X_t via $\iota(X_t)\omega = dH_t$. The function H is called a time-dependent Hamiltonian.

Remark 6.17. Time-dependent Hamiltonian is determined by the Hamiltonian isotopy up to an additive function $c : [0,1] \to \mathbb{R}$. Is M is simply connected, then every closed 1-form is exact, thus every symplectic vector field is Hamiltonian, finally, every symplectic isotopy is Hamiltonian.

What happens if we are given a time-independent Hamiltonian?

Definition 6.18 (Hamiltonian flow)**.** Suppose (M, ω) is a closed symplectic manifold, given a time-independent Hamiltonian function H, the isotopy generated by its corresponding Hamiltonian vector field X_H is called the Hamiltonian flow associated to H.

Notice, the compactness of the manifold in the definition is only added to guarantee the existence of global flow associated to the given Hamiltonian. It is totally fine to work without this assumption.

Remark 6.19. The identity $dH(X_H) = \omega(X_H, X_H) = 0$ shows that the Hamiltonian vector field X_H is tangent to the level sets $H^{-1}(c), c \in \mathbb{R}$.

Example 6.20. In the last talk, we mentioned that every orientable surface admits a symplectic form. On $S^2 \setminus \{(0, \pm 1)\}$, in particular, the area form induced by the Euclidean metric on $\mathbb{R}^3$ can be written as $\omega = d\theta \wedge dx_3$, which is a symplectic form(see figure 8, the two figures below are borrowed from [MS17]). Consider the height function $H = x_3 : S^2 \setminus \{(0, \pm 1\} \to \mathbb{R}$. Then the level sets of H are exactly circles at constant height. The corresponding Hamiltonian vector field X_H is the one satisfying $d\theta \wedge dx_3(X_H, \cdot) = dx_3$, which is obviously $\frac{\partial}{\partial \theta}$. Thus, the Hamiltonian flow Φ_H^t is the rotation of the sphere about its vertical axis at constant angular speed.

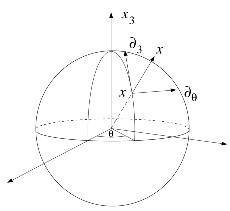


FIGURE 8. Polar coordinates (θ, x_3)

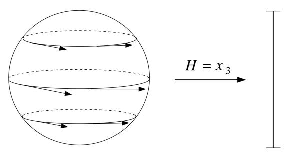


FIGURE 9. Rotating

Remark 6.21. In this example, we can see that the motion of a point along the Hamiltonian flow is within the level set, namely, preserves H. This is the familiar law of conservation of energy. Actually, the initial motivation for symplectic geometry lies exactly in the study of classical mechanical systems, such as the planetary system. Under that context, the Hamiltonian function is the sum of kinetic energy and potential energy. Then by the law of conservation of energy, any motion of a planet preserves the total energy of the system, which is just like we did before.

D. Hamiltonian symplectomorphism

Definition 6.22 (Hamiltonian symplectomorphism)**.** A symplectomorphism is called Hamiltonian if there exists a Hamiltonian isotopy $\Psi_t \in \mathrm{Symp}(M, \omega)$ from $\Psi_0 = \mathrm{id}$ to $\Psi_1 = \Psi$. We denote the space of Hamiltonian symplectomorphisms by $\mathrm{Ham}(M, \omega)$.

In the case M is not compact, a Hamiltonian function not necessarily generates a global isotopy, thus, it is also useful to consider a compactly supported Hamiltonian function.

Definition 6.23 (compactly supported Hamiltonian symplectomorphsim)**.** Let $H : [0,1] \times M \to \mathbb{R}$ be a compactly supported Hamiltonian, which determines a compactly supported Hamiltonian isotopy $\{\Psi_t\}_{0 \leq t \leq 1}$ and its time-1 map will be denoted by $\Phi_H := \Psi_1$. Every such Hamiltonian symplectomorphism is called compactly supported, and we denote $Ham_c(M, \omega) := \{\Phi_H | H \in C_0^\infty([0,1] \times M)\}$.

Remark 6.24. If $\{\Psi_t\}_{0\leq t\leq 1}$ is a Hamiltonian isotopy, then Ψ_t is Hamiltonian for any $t \in [0,1]$. This can be shown by re-parameterization, namely consider $\Phi_s := \Psi_{st}$ for $\forall s \in [0,1]$. Hence, by definition $\mathrm{Ham}(M)$ is path-connected, and therefor a subset of $\mathrm{Symp}_0(M)$.

Actually, $Ham(M)$ is a normal subgroup of $\mathrm{Symp}(M)$, to show that, we will need the following technical lemma.

Lemma 6.25. *(i) Let M be a smooth manifold, $\{\Phi_t\},\{\Psi_t\} \subset \mathrm{Diff}(M)$ be two isotopies, then*

$$\partial_t(\Phi_t \circ \Psi_t) = (\partial_t\Phi_t) \circ \Psi_t + (d\Phi_t) \circ \partial_t\Psi_t.$$

(ii) Let (M,ω) be a symplectic manifold. For every Hamiltonian function $H : M \to \mathbb{R}$ and every symplectomorphism $\Psi \in \mathrm{Symp}(M,\omega)$ we have $\Psi^*X_H = X_{H\circ\Psi}$, $\Psi_*X_H = X_{H\circ\Psi^{-1}}$.

Proposition 6.26. $\mathrm{Ham}(M)$ *is a path-connected normal subgroup of* $\mathrm{Symp}(M,\omega)$.

Proof. Given any $\Phi,\Psi \in \mathrm{Ham}(M)$, then there will be Hamiltonian function $F_t, G_t : [0,1] \times M \to \mathbb{R}$, and corresponding Hamiltonian isotopies $\{\Phi_t\}_{0\leq t\leq 1}, \{\Psi_t\}_{0\leq t\leq 1}$, which join id to Φ,Ψ respectively, that is, $\frac{d}{dt}\Phi_t = X_{F_t} \circ \Phi_t$, $\frac{d}{dt}\Psi_t = X_{G_t} \circ \Psi_t$.

To prove that $\Phi \circ \Psi$ is Hamiltonian, consider the isotopy: $\Phi_t \circ \Psi_t$, then

$$\begin{aligned}\frac{d}{dt}(\Phi_t \circ \Psi_t) &= (\frac{d}{dt}\Phi_t) \circ \Psi_t + (d\Phi_t) \circ (\frac{d}{dt}\Psi_t)\\ &= X_{F_t} \circ \Phi_t \circ \Psi_t + (d\Phi_t) \circ X_{G_t} \circ \Psi_t\\ &= X_{F_t} \circ \Phi_t \circ \Psi_t + (\Phi_t)_*(X_{G_t}) \circ \Phi_t \circ \Psi_t\\ &= X_{F_t + G_t\circ\Phi_t^{-1}} \circ \Phi_t \circ \Psi_t.\end{aligned}$$

Hence, $\Phi_t \circ \Psi_t$ is a Hamiltonian isotopy with $F_t + G_t \circ \Phi_t^{-1}$ as its Hamiltonian function. So, $\Phi \circ \Psi \in \mathrm{Ham}(M)$.

To prove that Φ^{-1} is Hamiltonian, consider isotopy Φ_t^{-1},

$$\begin{aligned}0 = \frac{d}{dt}(\Phi_t \circ \Phi_t^{-1}) &= (\frac{d}{dt}\Phi_t) \circ \Phi_t^{-1} + (d\Phi_t) \circ (\frac{d}{dt}\Phi_t^{-1})\\ &= X_{F_t} \circ \Phi_t \circ \Phi_t^{-1} + d\Phi_t(\frac{d}{dt}\Phi_t^{-1})\end{aligned}$$

therefore,

$$\frac{d}{dt}\Phi_t^{-1} = d\Phi_t^{-1}(X_{-F_t}) = (\Phi_t)^*(X_{-F_t}) \circ \Phi_t^{-1} = X_{-F_t\circ\Phi_t} \circ \Phi_t^{-1}.$$

Thus, Φ_t^{-1} is a Hamiltonian isotopy with $-F_t \circ \Phi_t$ as its Hamiltonian function, so $\Phi^{-1} \in \mathrm{Ham}(M)$.

Finally, to show that $\mathrm{Ham}(M)$ is a normal subgroup, let χ be a symplectomorphism,

$$\begin{aligned}\frac{d}{dt}(\chi^{-1} \circ \Phi_t \circ \chi) &= d\chi^{-1} \circ (\frac{d}{dt}\Phi_t) \circ \chi = d\chi^{-1} \circ X_{F_t} \circ \Phi_t \circ \chi\\ &= \chi^*(X_{F_t}) \circ \chi^{-1} \circ \Phi_t \circ \chi = \chi_{F_t\circ\chi} \circ (\chi^{-1} \circ \Phi_t \circ \chi).\end{aligned}$$

That tells us $\chi^{-1} \circ \Phi_t \circ \chi$ is Hamiltonian with Hamiltonian function $F_t \circ \chi$ and therefore, $\mathrm{Ham}(M)$ is normal. □

Theorem 6.27 ([Ban78, Ban97]). *The group* $\mathrm{Ham}(M)$ *is perfect and simple for closed symplectic manifolds M.*

Talk 7: The Flux homomorphism

Yilin, Patrik

We would like to characterize Hamiltonian isotopies. In the first step we look at the relative simple case where the symplectic form is exact.

Definition 7.1. A symplectic manifold (M, ω) is *exact* if ω is exact, i.e. there exists an one-form λ such that $\mathrm{d}\lambda = \omega$.

Proposition 7.2. *Let (M, ω) be an exact symplectic manifold, $\phi_t \in \mathrm{Diff}(M)$ an isotopy starting from the identity $\phi_0 = \mathrm{id}$. Then ϕ_t is a Hamiltonian isotopy is and only if $\phi_t^*\lambda - \lambda$ is exact, i.e.*

$$\phi_t^*\lambda - \lambda = \mathrm{d}F_t$$

for some smooth family of functions F_t. The function F_t is given by

$$F_t = \int_0^t (\iota_{X_s}\lambda + H_s) \circ \phi_s \mathrm{d}s$$

Proof. Let ϕ_t be a Hamiltonian isotopy, generated by the vector field $X_t : M \to TM$. Then by the differential rule given in proposition 6.14 we have

$$\partial_t(\phi_t^*\lambda - \lambda) = \partial_t\phi_t^*\lambda \overset{6.14}{=} \phi_t^*\mathcal{L}_{X_t}\lambda \overset{Cartan}{=} \phi_t^*(\iota_{X_t}\mathrm{d}\lambda + \mathrm{d}(\iota_{X_t}\lambda)) = \phi_t^*\mathrm{d}(\iota_{X_t}\lambda + H_t) = \partial_t \mathrm{d}F_t.$$

The one-forms differ only by a constant, but they agree on $t = 0$, so the equality holds. Conversely, we construct H_t:

$$H_t = -\iota_{X_t}\lambda + (\partial F_t) \circ \phi_t^{-1}.$$

Then

$$0 = \partial_t(\phi_t^* - \mathrm{d}F_t) = \phi_t^*(\iota_{X_t}\mathrm{d}\lambda + \mathrm{d}(\iota_{X_t}\lambda)) + \mathrm{d}\partial_t F^t = \phi_t^*(\iota_{X_t}\mathrm{d}\lambda - \mathrm{d}H_t)$$

Hence $\iota_{X_s}\omega = \mathrm{d}H_s$ □

Remark 7.3. A symplectomorphism ϕ of an exact symplectic manifold $(M, \omega = \mathrm{d}\lambda)$ is called exact (with respect to λ) if $\phi^*\lambda - \lambda$ is exact.

Corollary 7.4. *Let (M, t) be an exact symplectic manifold and ϕ_t a symplectic isotopy. Then ϕ_t is Hamiltonian for each t if and only if ϕ_t is an Hamiltonian isotopy. The proposition above shows that every Hamiltonian symplectomorphism is exact with repect to any (reasonable) 1-form.*

Proof. Clearly we can choose F_t smoothly depending on ϕ_t. □

After understanding the simple case, we would like to characterize Hamiltonian isotopies in a more general setting where the manifold is not necessarily exact. For that we need the notion of flux homomorphism $\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0 \to H^1(M)$. In the following we understand an element in the universal cover of $\mathrm{Symp}(M, \omega)$ as the homotopy class $\{\psi_t\}$ from id to ψ, and the group structure on the universal cover is given by concatenating.

Definition 7.5. The flux homomorphism $\mathrm{Flux} : \widetilde{\mathrm{Symp}}_0 \to H^1(M)$ is defined by

$$\mathrm{Flux}(\{\psi_t\}) : \int_0^1 [\iota_{X_t}\omega]\mathrm{d}t \in H^1(M; \mathbb{R})$$

where X_t is the unique generating vector field satisfying $\partial_t\psi_t = X_t \circ \psi_t$

We need to check the right-hand side is well-defined.

Lemma 7.6 (Well-definedness)**.** *The right-hand side depends only on the homotopy class of ψ_t.*

Proof. We identify $H^1(M; \mathbb{R})$ with $\mathrm{Hom}(\pi_1(M); \mathbb{R})$ by the map

$$[\omega] \to ([\gamma] \mapsto \int_\gamma \omega).$$

Then the cohomology class in the definition corresponds to the homomorphism $\pi_1(M) \to \mathbb{R}$ defined by

$$\gamma \mapsto \int_0^1 \int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma})\mathrm{d}s\mathrm{d}t$$

for representative $\gamma : \mathbb{R}/\mathbb{Z} \to M$. Since ϕ_t are symplectomorphisms, the 1-forms $\iota_{X_t}\omega$ are closed. So the integral (by Stokes' theorem) depends only on the homotopy class of γ.

Define $\beta : \mathbb{R}/\mathbb{Z}$ by $\beta(s,t) := \psi_t^{-1}(\gamma(s))$. Geometrically β is a cylinder in M starting with γ and flows along the isotopy $\{\psi_t^{-1}\}$. Differentiate both sides of $\psi_t(\beta(s,t)) = \gamma(s)$ with respect to s and t we get

$$d\psi_t(\beta)\frac{\partial \beta}{\partial s} = \dot{\gamma}(s), \quad d\psi_t(\beta)\frac{\partial \beta}{\partial t} + X_t(\gamma(s)) = 0$$

Since $\psi_t^*\omega \overset{Def}{=} \omega$, we have

$$\mathrm{Flux}(\{\psi_t\})(\gamma) = \int_0^1 \int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma})\mathrm{d}s = \int_0^1 \int_0^1 \omega(\frac{\partial \beta}{\partial s}, \frac{\partial \beta}{\partial t})\mathrm{d}s = \int_{\mathbb{R}/\mathbb{Z}\times[0,1]} \beta^*\omega.$$

The integral (in fact by naturality of pullback) depends only on the homotopy class of β and hence of ψ, we can prove with following argument:

Define a homotopy $u \to \{\psi_t^u\}$ be a homotopy fixing the start point id map and end point $\psi := \psi_1^u$. Define $\Gamma : u \to \beta_u := (\psi_t^u)^{-1}(\gamma(s))$ to be a homotopy. For convenience we take $X := \mathbb{R}/\mathbb{Z} \times [0,1] = S^1 \times [0,1]$, then

$$\int_X \beta_1^*\omega - \int_X \beta_0^*\omega = \int_{X\times\{1\}} \Gamma^*\omega - \int_{X\times\{0\}} \Gamma^*\omega \overset{Stokes}{=} \int_{X\times I} d(\Gamma^*\omega) = \int_{X\times I} \Gamma^*(d\omega) = 0$$

□

Remark 7.7. The result is the same if we use the more natural map $\beta(s,t) := \psi_t(\gamma(s))$. The flux is geometrically the symplectic area swept by γ under the isotopy.

Theorem 7.8 ([MS17], Flux characterizes Hamiltonian isotopies)**.** *Let (M,ω) be a closed connected symplectic manifold and $\psi \in \mathrm{Symp}_0(M,\omega)$. Then ψ is a Hamiltonian symplectomorphism $\Leftrightarrow$ there exists a symplectic isotopy*

$$[0,1] \to \mathrm{Symp}_0(M,\omega)$$
$$t \mapsto \psi_t$$

such that

$$\psi_0 = \mathrm{id}, \qquad \psi_1 = \psi, \qquad \mathrm{Flux}(\{\psi_t\}) = 0.$$

Moreover, if $\mathrm{Flux}(\{\psi_t\}) = 0$ *then* $\{\psi_t\}$ *is isotopic with fixed endpoints to a Hamiltonian isotopy.*

Proof. If ψ is Hamiltonian, it is the endpoint of a Hamiltonian isotopy ψ_t corresponding to some family of Hamiltonian functions $H_t : M \to \mathbb{R}$, and

$$\mathrm{Flux}(\{\psi_t\}) = \int_0^1 [\iota(X_t)\omega]\, dt = \int_0^1 [\mathrm{d}H_t]\, dt = 0.$$

Conversely, let $\psi_t \in \mathrm{Symp}_0(M,\omega)$ be a symplectic isotopy from $\psi_0 = \mathrm{id}$ to $\psi_1 = \psi$ s.t. $\mathrm{Flux}(\{\psi_t\}) = 0$, and define $X_t \in \mathcal{X}(M,\omega)$ by

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

We know that the integral $\int_0^1 \iota(X_t)\omega\, dt$ is exact, and what we must do is change the isotopy ψ_t so that $\iota(X_t)\omega$ is exact for each t. Equivalently, we must make the integral $\int_0^T \iota(X_t)\omega\, dt$ exact for each $T \in [0,1]$. Thus, $\mathrm{Flux}(\{\psi_t\}_{0\le t\le T}) = 0$ for every $T \in [0,1]$. The first step is to modify ψ_t by a Hamiltonian isotopy s.t. the 1-form $\int_0^T \iota(X_t)\omega\, dt$ is 0 rather than merely exact. To achieve this, note first that since $\mathrm{Flux}(\{\psi_t\}) = 0$, there exists a function $F : M \to \mathbb{R}$ s.t.

$$\int_0^1 \iota(X_t)\omega\, dt = dF.$$

Let ϕ_F^s be the Hamiltonian flow of F. Since ϕ_F^s is Hamiltonian for each $s \in \mathbb{R}$, it suffices to prove the Theorem for the composition $\phi_F^{-1} \circ \psi$ instead of ψ. But this is the endpoint of the juxtaposition $\psi_t^{'}$ defined by $\psi_t^{'} := \psi_{2t}$ for $0 \le t \le \frac{1}{2}$, and $\psi_t^{'} := \phi_F^{1-2t} \circ \psi_1$ for $\frac{1}{2} \le t \le 1$. This isotopy $\psi_t^{'}$ (or a suitable smooth reparametrization) is generated by a smooth family of vector fields $X_t^{'}$ s.t. $\int_0^1 \mathrm{X}_t^{'}\, dt = 0$. Hence, from now on we assume that $\psi = \psi_1$ for some isotopy with

$$\int_0^1 \mathrm{X}_t \, dt = 0.$$

Next, for every t, let $\theta_t^s \in \mathrm{Symp}_0(M, \omega)$, $s \in \mathbb{R}$, be the flow generated by the symplectic vector field

$$Y_t := -\int_0^t \mathrm{X}_\lambda \, d\lambda.$$

Thus,

$$\partial_s \theta_t^s = Y_t \circ \theta_t^s, \qquad \theta_t^0 = \mathrm{id}\,.$$

Observe that $Y_0 = Y_1 = 0$ and hence $\theta_0^s = \theta_1^s = \mathrm{id}$ for all s.
We claim that

$$\phi_t := \theta_t^1 \circ \psi_t$$

is the desired Hamiltonian isotopy from $\phi_0 = \mathrm{id}$ to $\phi_1 = \psi_1 = \psi$. To see this, note that because Flux is a homomorphism of groups,

$$\begin{aligned} \mathrm{Flux}(\{\phi_t\}_{0 \le t \le T}) &= \mathrm{Flux}(\{\theta_t^1\}_{0 \le t \le T}) + \mathrm{Flux}(\{\psi_t\}_{0 \le t \le T}) \\ &= \mathrm{Flux}(\{\theta_T^s\}_{0 \le s \le 1}) + \int_0^T [\iota(X_t)\omega] \; dt \\ &= [\iota(Y_T)\omega] + \int_0^T [\iota(X_t)\omega] \; dt \\ &= 0. \end{aligned}$$

Here the second step uses the homotopy invariance of the flux, and the third follows from the fact that θ_T^s is the flow of Y_T. □

Proposition 7.9 ([MS17], Exact sequences induced by flux)**.** *Let (M, ω) be a closed, connected symplectic manifold. Then:*

(1) *There is an exact sequence of simply connected Lie groups*

$$0 \to \widetilde{\mathrm{Ham}}(M, \omega) \to \widetilde{\mathrm{Symp}}_0(M, \omega) \to H^1(M; \mathbb{R}) \to 0,$$

where $\widetilde{\mathrm{Ham}}(M, \omega)$ is the universal cover of $\mathrm{Ham}(M, \omega)$ and the third arrow is the flux homomorphism.

(2) *There is an exact sequence of Lie algebras*

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathcal{X}(M, \omega) \to H^1(M; \mathbb{R}) \to 0.$$

Here the third map is $H \mapsto X_H$ and the fourth map is $X \mapsto [\iota_X \omega]$.

(3) *There is an exact sequence of groups*

$$0 \to \pi_1(\mathrm{Ham}(M, \omega)) \to \pi_1(\mathrm{Symp}_0(M, \omega)) \to \Gamma_\omega \to 0,$$

where Γ_ω is the flux group.

(4) *There is an exact sequence of groups*

$$0 \to \mathrm{Ham}(M, \omega) \to \mathrm{Symp}_0(M, \omega) \xrightarrow{\rho} H^1(M; \mathbb{R})/\Gamma_\omega \to 0,$$

where ρ is the map induced by Flux. Thus, $\mathrm{Symp}_0(M, \omega)/\,\mathrm{Ham}(M, \omega)$ is isomorphic to $H^1(M; \mathbb{R})/\Gamma_\omega$.

Proof. From Proposition 10.2.12 ([MS17]) we know that every smooth path $\psi_t \in \mathrm{Ham}(M)$ which starts at the identity is a Hamiltonian isotopy and hence has zero flux, i.e $\mathrm{Flux}(\{\psi_t\}) = 0$. This shows that $\widetilde{\mathrm{Ham}}(M) \subset \ker(\mathrm{Flux})$. Conversely, Theorem 7.8 shows that if $\mathrm{Flux}(\{\psi_t\}) = 0$ then the path ψ_t is homotopic, with fixed endpoints, to a Hamiltonian isotopy, and hence $\{\psi_t\} \in \widetilde{\mathrm{Ham}}(M)$. This shows that $\widetilde{\mathrm{Ham}}(M) \supset \ker(\mathrm{Flux})$. We conclude

$$\widetilde{\mathrm{Ham}}(M) = \ker(\mathrm{Flux})$$

The first statement now follows from the fact that Flux is surjective. The second statement is obvious. For the third statement we have to show that $\pi_1(\mathrm{Ham}(M))$ injects into $\pi_1(\mathrm{Symp}_0(M))$. To see this, it is enough to show that any path $[0,1] \to \widetilde{\mathrm{Symp}_0}(M)$ with endpoints in $\widetilde{\mathrm{Ham}}(M)$ is isotopic with fixed endpoints to a path in $\widetilde{\mathrm{Ham}}(M) = \ker(\mathrm{Flux})$ This is a parametrized version of the first statement. The last statement is obvious. □

Talk 8: The Calabi homomorphism and Calabi quasimorphisms

Moritz, Arthur

We recall from the last section that the flux induces an exact sequence

$$0 \to \mathrm{Ham}(M,\omega) \to \mathrm{Symp}_0(M,\omega) \overset{\rho}{\to} H^1(M;\mathbb{R})/\Gamma_\omega \to 0,$$

for a closed, symplectic manifold (M,ω). Banyaga [Ban78] proved that if (M,ω) is closed and symplectic, the group $\mathrm{Ham}(M,\omega)$ is simple. We now want to construct a similar sequence for exact manifolds $(M,\omega = d\lambda)$. In the following we will define the Calabi homomorphism $\mathrm{CAL} : \mathrm{Ham}_c(M,\omega) \longrightarrow \mathbb{R}$, which is non-trivial. Hence for exact manifolds (M,ω), the group $\mathrm{Ham}_c(M,\omega)$ is not simple, but Banyaga [Ban78] proved that the kernel of CAL is simple.

Consider an exact symplectic manifold (M,ω), for which $\omega = d\lambda$. In this case, M cannot be closed, because if it was,

$$vol(M) = \int_M \omega^n = \int_M d(\lambda \wedge \omega^{n-1}) = \int_{\delta M} \lambda\omega^{n-1} = 0, \tag{10}$$

contradicting the fact that ω^n is a volume form.

By Proposition 7.2, given a compactly supported Hamiltonian diffeomorphism ϕ of (M,ω), where $\omega = d\lambda$, there exists a unique compactly supported function $F : M \longrightarrow \mathbb{R}$ such that $\phi^*\lambda - \lambda = dF$.

We define

Definition 8.1. $\mathrm{CAL}(\phi) = \frac{1}{n+1}\int_M F\omega^n$.

This number, however, depends , on the choice of the primitive λ for the symplectic form ω. We derive another formula for this quantity that does not depend on the choice of λ:

Lemma 8.2. $\mathrm{CAL}(\phi) = \int_0^1 \int_M H_t\omega^n dt$.

Proof. By Proposition 7.2, there is a unique compactly supported family F_t such that $\phi_t^*\lambda - \lambda = dF_t$, given by

$$F_t = \int_0^t \phi_s^*(H_s + \iota_{X_s}\lambda)ds. \tag{11}$$

Since ϕ_t preserves ω, it follows that

$$\begin{aligned}\int_M \frac{d}{dt}F_t\omega^n &= \int_M \phi_t^*(H_t + \iota_{X_t}\lambda)\omega^n \\ &= \int_M (H_t + \iota_{X_t}\lambda)\omega^n.\end{aligned} \tag{12}$$

Then, we compute

$$\begin{aligned}\int_0^1\int_M H_t\omega^n dt &= \frac{1}{n+1}\int_0^1\int_M (nH_t - \iota_{X_t}\lambda + H_t + \iota_{X_t\lambda})\omega^n dt \\ &= \frac{1}{n+1}\int_0^1\int_M \left(\frac{d}{dt}F_t + nH_t - \iota_{X_t}\lambda\right)\omega^n dt \\ &= \frac{1}{n+1}\int_M F_1\omega^n + \frac{1}{n+1}\int_0^1\int_M (nH_t - \iota_{X_t}\lambda)\omega^n dt \\ &= \mathrm{CAL}(\phi) + \frac{1}{n+1}\int_0^1\int_M (nH_t - \iota_{X_t}\lambda)\omega^n dt.\end{aligned} \tag{13}$$

To show that the last term is zero, we consider the form $\omega^n \wedge \lambda$, which is zero since $\dim M = 2n$.

$$\begin{aligned} 0 &= \iota_{X_t}(\lambda \wedge \omega^n) \\ &= (\iota_{X_t}\lambda)\omega^n - \lambda \wedge \iota_{X_t}(\omega^n) \\ &= (\iota_{X_t}\lambda)\omega^n - n\lambda \wedge \iota_{X_t}\omega \wedge \omega^{n-1} \\ &= (\iota_{X_t}\lambda)\omega^n - ndH_t \wedge \lambda \wedge \omega^{n-1} \\ &= (\iota_{X_t}\lambda)\omega^n - nd(H_t\lambda) \wedge \omega^{n-1} - nH_t d\lambda \wedge \omega^{n-1} \\ &= (\iota_{X_t}\lambda - nH_t)\omega^n + nd(H_t\lambda \wedge \omega^{n-1}). \end{aligned} \tag{14}$$

Hence, for every t,

$$\int_M (\iota_{X_t}\lambda - nH_t)\omega^n = 0, \tag{15}$$

giving the proposed result. □

With this, we get a well defined map

$$\mathrm{CAL} : \mathrm{Ham}_c(M, \omega) \longrightarrow \mathbb{R}, \tag{16}$$

when $\omega = d\lambda$.

Theorem 8.3. CAL *is a homomorphism, called the Calabi homomorphism.*

Proof. Let ϕ and ψ be Hamiltonian diffeomorphisms. Then,

$$\begin{aligned} (\psi \circ \phi)^*\lambda - \lambda &= \phi^*(\psi^*\lambda - \lambda) + (\phi^*\lambda - \lambda) \\ &= \phi^* dG + dF, \end{aligned} \tag{17}$$

by Proposition 7.2. Hence,

$$\begin{aligned} \mathrm{CAL}(\psi \circ \phi) &= \frac{1}{n+1}\int_M (G \circ \phi)\omega^n + \frac{1}{n+1}\int_M F\omega^n \\ &= \mathrm{CAL}(\psi) + \mathrm{CAL}(\phi), \end{aligned} \tag{18}$$

since ϕ preserves ω. □

Lemma 8.4. CAL *is continuous and its kernel is simple.*

Proof. See [Ban97] for a proof. □

The next lemma shows that CAL is indeed unique up to rescaling.

Lemma 8.5. *For M an exact manifold without boundary,* CAL *is the only continuous homomorphism from* $\mathrm{Ham}_c(M, \omega)$ *to the real numbers, up to rescaling.*

Proof. Let K be the kernel of CAL. By Lemma 8.2, the Calabi homomorphism is surjective, so CAL induces an isomorphism $\overline{\mathrm{CAL}} : \mathrm{Ham}_c(M, \omega)/K \longrightarrow \mathbb{R}$. Given another continuous homomorphism f, $K \cap Kerf$ a normal subgroup of K, which is hence trivial or equal to K itself, since K is simple by Lemma 8.4. If $K \cap Kerf$ is trivial, the map

$$f \oplus \mathrm{CAL} : \mathrm{Ham}_c(M, \omega) \longrightarrow \mathbb{R} \oplus \mathbb{R} \tag{19}$$

would be injective, contradicting the fact that the Hamiltonian group is non-abelian. If $\mathrm{Ker}\, f \supset K$ then f also factors through a homomorphism $\overline{f}$ on $\mathrm{Ham}_c(M, \omega)/K$. The composition $\overline{f} \circ \overline{\mathrm{CAL}}^{-1}$ is a continuous automorphism of $\mathbb{R}$, which is a multiplication by a constant. □

A. Calabi-homomorphism on non-exact symplectic manifolds

In this section, (M, ω) is no longer supposed to be exact. Furthermore, M shall be non-compact and without border. Take $\varphi \in \mathrm{Ham}_c(M, \omega)$, let $(\varphi_t)_{t\in[0,1]}$ be a compactly supported Hamiltonian isotopy from id to φ and H_t the unique corresponding compactly supported Hamiltonian. Then in general $\int_0^1 \int_M H_t \omega^n dt$ is not independent of the choice of $(\varphi_t)_{t\in[0,1]}$. Hence, the Calabi homomorphism cannot be defined on $\mathrm{Ham}_c(M)$ like in the exact case. However, a natural generalisation is to define it on $\widetilde{\mathrm{Ham}_c(M)}$ instead:

Proposition 8.6. *The map*

$$\begin{aligned} \widetilde{\mathrm{CAL}} : \widetilde{\mathrm{Ham}_c(M)} &\to \mathbb{R} \\ (\varphi_t)_{t\in[0,1]} &\mapsto \int_0^1 \int_M H_t \omega^n dt \end{aligned} \tag{20}$$

is a well-defined continuous and surjective homomorphism.

At some point in the proof we will need the following lemma, which is somehow a generalization of the theorem of Schwarz.

Lemma 8.7. *Let $h_{s,t}$ be a smooth family in $\mathrm{Diff}_0(M)$ with $h_{0,0} = \mathrm{id}$. Define for all $x \in M$, $X_{s,t}(x) = \frac{dh_{s,t}}{dt}(h_{s,t}^{-1}(x))$ and $Y_{s,t}(x) = \frac{dh_{s,t}}{ds}(h_{s,t}^{-1}(x))$. Then*

$$\frac{dX_{s,t}}{ds} = \frac{dY_{s,t}}{dt} + [X_{s,t}, Y_{s,t}]. \tag{21}$$

Now let us move on to the proof of the proposition.

Proof. First, define the linear map

$$c : X_H \mapsto \int_m H\omega^n, \tag{22}$$

where X_H is a Hamiltonian vector field and H is the unique compactly supported Hamiltonian associated to X_H. We will show that c is a Lie-algebra homomorphism, i.e. for all Hamiltonian vector fields X and Y, $c([X, Y]) = 0$, since $\mathbb{R}$ is abelian. In fact,

$$\mathcal{L}_X i(Y)\omega = i(Y)\mathcal{L}_X\omega + i([X, Y])\omega = i([X, Y])\omega \tag{23}$$

since X is a symplectic vector field and

$$\mathcal{L}_X i(Y)\omega = d(i(X)i(Y)\omega) \tag{24}$$

by Cartan's formula, since $i(Y)\omega$ is exact. Hence, the Hamiltonian $H_{[X,Y]}$ associated to $[X, Y]$ is given by $i(X)i(Y)\omega$. Finally,

$$\begin{aligned} i(X)(i(Y)\omega \wedge \omega^n) &= (i(X)i(Y)\omega)\omega^n - i(Y)\omega \wedge i(X)\omega^n \\ &= H_{[X,Y]}\omega^n - n(dH_Y \wedge dH_X \wedge \omega^{n-1}) = 0, \end{aligned} \tag{25}$$

since M is of dimension n. Hence, $H_{[X,Y]}\omega^n = d(nH_Y \wedge dH_X \wedge \omega^{n-1})$. Therefore, Stokes yields that $c([X, Y]) = 0$.

Let us now prove that $\widetilde{\mathrm{CAL}}$ is well-defined. Take $h_1 \in \mathrm{Ham}_c(M)$ and let $(h_t)_{t\in[0,1]}$ and $(\bar{h}_t)_{t\in[0,1]}$ be two isotopies from id to h_1 such that there exists a smooth 2-parameter homotopy with fixed borders $h : [0, 1]^2 \to \mathrm{Ham}_c(M)$ such that $h(t, 0) = h_t$ and $h(t, 1) = \bar{h}_t$ for all $t \in [0, 1]$. Define $X_{s,t}$ and $Y_{s,t}$ like in Lemma 8.7. Notice that c commutes with $\frac{d}{dt}$ and $\frac{d}{ds}$, since for all Y we have

$$\begin{aligned} d(\frac{d}{ds}H_{X_{s,z}})(Y) &= \frac{d}{ds}dH_{X_{s,z}}(Y) = \frac{d}{ds}\omega(X_{s,t}, Y) \\ &= \omega(\frac{d}{ds}X_{s,t}, Y) = dH_{\frac{d}{ds}X_{s,t}}(Y). \end{aligned} \tag{26}$$

Hence, Lemma 8.7 yields

$$\begin{aligned}\frac{d}{ds}\int_0^1 c(X_{s,t})dt &= \int_0^1 c(\frac{d}{ds}X_{s,t}) \\ &= \int_0^1 c(\frac{d}{dt}Y_{s,t}) + \int_0^1 c([X_{s,t}, Y_{s,t}]) \\ &= \int_0^1 c(\frac{d}{dt}Y_{s,t}) = c(Y_{s,1}) - c(Y_{s,0}),\end{aligned} \tag{27}$$

since c vanishes on commutators. But now $Y_{s,1} = Ys,0 = 0$ since h is a homotopy with fixed border. It follows that $\frac{d}{ds}\int_0^1 c(X_{s,t})dt = 0$ and thus,

$$\widetilde{\mathrm{CAL}}((h_t)) = \int_0^1 c(X_{0,t})dt = \int_0^1 c(X_{1,t})dt = \widetilde{\mathrm{CAL}}((\bar{h}_t)). \tag{28}$$

Therefore, $\widetilde{\mathrm{CAL}}$ is well-defined on $\widetilde{\mathrm{Ham}_c}(M)$. The surjectivity and the homomorphism property follow from arguments similar to those used to prove the same properties for the flux homomorphism. ☐

Now let Λ denote the image of $\pi^1(\mathrm{Ham}_c(M)) \subset \widetilde{\mathrm{Ham}_c}(M)$ under $\widetilde{\mathrm{CAL}}$. The snake lemma implies that $\widetilde{\mathrm{CAL}}$ induces a homomorphism

$$\mathrm{CAL} : \mathrm{Ham}_c(M) \to \mathbb{R}/\Lambda,$$

the Calabi homomorphism.

Remark 8.8. In general, it is hard to compute Λ explicitly, however, two "extreme" cases are well-studied. As we have seen before, Λ is just trivial when (M, ω) is exact. On the other hand, we shall see in the next section that in the case where M is closed - which has been excluded here - $\mathrm{Ham}_c(M)$ is simple and hence $\Lambda = \mathbb{R}$.

Let us end this section with a result due to Banyaga.

Theorem 8.9. *The kernel of* CAL *is simple.*

B. On the group of Hamiltonian symplectomorphisms: simplicity, normal subgroups and quasi-morphisms

Let (M, ω) be a symplectic manifold. In the following section, we will study the group of Hamiltonian symplectomorphisms of M in the two "extreme" cases mentioned before. To justify the distinction between exact and compact recall that:

Proposition 8.10. *Let (M, ω) be an exact symplectic manifold. Then M is not closed.*

Proof. Let $n \in \mathbb{N}$ such that M is of dimension $2n$ and take $\lambda \in \Omega^1(M)$ such that $\omega = d\lambda$. Then

$$d(\lambda \wedge \omega^{n-1}) = d\lambda \wedge \omega^{n-1} + \lambda \wedge d(\omega^{n-1}) = \omega^n, \tag{29}$$

since ω is closed. Hence, ω^n is exact. Suppose now that M is compact without border. We know that ω^n defines a volume form on M. Therefore, Stokes' theorem yields a contradiction. ☐

Recall further that we have a short exact sequence

$$0 \to \widetilde{\mathrm{Ham_c}}(M, \omega) \to \widetilde{\mathrm{Symp_{c,0}}}(M, \omega) \to H^1_c(M; \mathbb{R}) \to 0, \tag{30}$$

where the third arrow is the flux homomorphism. Furthermore, if $\Gamma_\omega := Flux(\pi_1(\mathrm{Symp}_{c,0}(M, \omega))) \subset H^1(M; \mathbb{R})$ denotes the flux group, the previous short exact sequence induces the short exact sequence of groups

$$0 \to \mathrm{Ham_c}(M, \omega) \to \mathrm{Symp_{c,0}}(M, \omega) \to H^1_c(M; \mathbb{R})/\Gamma_\omega \to 0. \tag{31}$$

Remark 8.11. The flux group Γ_ω is generally even harder to construct then the group Λ from the previous section. However, it has been proven that Γ_ω is a discrete subgroup of $H^1_c(M; \mathbb{R})$ if and only if $\mathrm{Ham_c}(M, \omega)$is locally connected.

Let us next study what these short exact sequences become when (M, ω) is exact.

B.1. The exact case

In the following let (M,ω) denote an exact symplectic manifold and take $\lambda \in \Omega^1(M)$ such that $\omega = d\lambda$.

Lemma 8.12. *The map*

$$\begin{aligned} Flux : \widetilde{\mathrm{Symp}}_{c,0} &\to H^1_c(M) \\ (\psi_t)_{t\in[0,1]} &\mapsto [\psi_1^*\lambda - \lambda] \end{aligned} \tag{32}$$

is well-defined, independent of λ *and coincides with the Flux morphism defined in ???. In particular, Flux factorizes to a surjective morphism*

$$\begin{aligned} Flux : \mathrm{Symp}_{c,0} &\to H^1_c(M) \\ \psi &\mapsto [\psi^*\lambda - \lambda] \end{aligned} \tag{33}$$

and yields a short exact sequence

$$0 \to \mathrm{Ham_c}(M,\omega) \to \mathrm{Symp_{c,0}}(M,\omega) \overset{Flux}{\to} H^1_c(M;\mathbb{R}) \to 0. \tag{34}$$

Proof. First, for all $\psi \in \mathrm{Symp}_{c,0}$, $d(\psi^*\lambda - \lambda) = \psi^* d\lambda - d\lambda = \psi^*\omega - \omega = 0$. Since ψ is supposed to be compactly supported, $\psi^*\lambda - \lambda$ is a compactly supported closed 1-form and $[\psi^*\lambda - \lambda]$ is indeed well-defined.

Now, let $(\psi_t)_{t\in[0,1]}$ be an isotopy in $\mathrm{Symp}_{c,0}$ starting at the identity and let $(X_t)_{t\in[0,1]})$ denote the corresponding variation vector-fields. Then

$$\psi_1^*\lambda - \lambda = \int_0^1 \frac{d}{dt}\psi_t^*\lambda dt = \int_0^1 \psi_t^*\mathcal{L}_{X_t}\lambda dt = \int_0^1 \psi_t^* i(X_t)\omega dt + d(\int_0^1 \psi_t^* i(X_t)\lambda dt), \tag{35}$$

by Cartan's formula. Hence,

$$[\psi_1^*\lambda - \lambda] = \int_0^1 [\psi_t^* i(X_t)\omega] dt, \tag{36}$$

and the map is well-defined. Now, let $\lambda' \in \Omega^1(M)$ be a closed 1-form. This time, (37) becomes

$$\psi_1^*\lambda - \lambda = d(\int_0^1 \psi_t^* i(X_t)\lambda dt). \tag{37}$$

Thus, $\psi_1^*\lambda - \lambda$ is exact. Now, if λ' also satisfies $d\lambda' = \omega$, $\lambda - \lambda'$ is closed. Thus $\psi_1^*(\lambda - \lambda') - (\lambda - \lambda') = (\psi_1^*\lambda - \lambda) - (\psi_1^*\lambda' - \lambda')$ is exact and $[\psi_1^*\lambda - \lambda] = [\psi_1^*\lambda' - \lambda']$. Furthermore, it follows that $\int_0^1 [\psi_t^* i(X_t)\omega] dt = \int_0^1 [i(X_t)\omega] dt$. Thus, Flux is independent of the choice of λ and coincides with the flux morphism defined in ???.

Since $[\psi_1^*\lambda - \lambda]$ depends only on ψ_1, Flux induces a morphism on $\mathrm{Symp}_{c,0}$. Since we have a long exact sequence

$$0 \to \widetilde{\mathrm{Ham}}_{\mathrm{c}}(M,\omega) \to \widetilde{\mathrm{Symp}}_{\mathrm{c},0}(M,\omega) \to H^1_c(M;\mathbb{R}) \to 0,$$

the induced homomorphism is surjective. □

Hence, in the exact case Λ and Γ_ω are trivial.

Example 8.13 (The disc D^2)**.** Let ω be a symplectic form on D^2. Since D^2 is contractible, $H^2_c(D^2,\mathbb{R}) = 0$ and thus ω is exact. Furthermore, $H^1_c(D^2,\mathbb{R}) = 0$. Hence, the short exact sequence (34) implies $\mathrm{Ham}_c(D^2,\omega) \simeq \mathrm{Symp}_{c,0}(D^2,\omega)$ and by definition $\mathrm{Symp}_{c,0}(D^2,\omega) = \mathrm{Diff}_0^\infty(D^2, \partial D^2, area)$.

Now, in Example 4.9 we have constructed a family of linearly independent homogeneous quasi-morphisms, $(\mathfrak{Sign}_{n,D^2})$. One can show that the signature quasi-morphism which is used to define the $\mathfrak{Sign}_{n,D^2}$ is a homomorphism on $P_2(D^2)$ and hence, $\mathfrak{Sign}_{2,D^2}$ is a continuous homomorphism itself. Thus Lemma 8.5 implies that $\mathfrak{Sign}_{2,D^2}$ is a multiple of the Calabi homomorphism on $\mathrm{Ham}_c(D^2,\omega)$ and it is then easy to show that $\mathfrak{Sign}_{2,D^2}$ actually is the Calabi homomorphism.

Finally, Theorem 8.9 yields that $\mathrm{Ker}(\mathfrak{Sign}_{2,D^2})$ is a simple subgroup of $\mathrm{Ham}_c(D^2,\omega)$. Since the restrictions of all $(\mathfrak{Sign}_{n,D^2})$ for $n > 2$ are non-trivial on $\mathrm{Ker}(\mathfrak{Sign}_{2,D^2})$, we have even constructed a countable family of non-trivial quasi-morphism on $\mathrm{Ker}(\mathfrak{Sign}_{2,D^2})$.

B.2. The compact case

Let (M, c) be a closed symplectic manifold. The following result immediately extinguishs all our hope to construct a Calabi homomorphism on $\mathrm{Ham}(M)$.

Theorem 8.14. *Let (M, c) be a closed symplectic manifold. Then $\mathrm{Ham}(M)$ is simple.*

Example 8.15 (The torus $\mathbb{T}^2$)**.** Let ω be a symplectic form on $\mathbb{T}^2$ (exists since $\mathbb{T}^2$ is orientable). This time one can show that the sequence (31) becomes

$$0 \to \mathrm{Ham}(\mathbb{T}^2, \omega) \to \mathrm{Symp}_0(\mathbb{T}^2, \omega) \to H^1(\mathbb{T}^2; \mathbb{R})/\mathbb{Z}^2 \to 0.$$

Theorem 8.14 yields that $\mathrm{Ham}(\mathbb{T}^2, \omega)$ is simple and hence, there exists no non-trivial homomorphism from $\mathrm{Ham}(\mathbb{T}^2, \omega)$ to $\mathbb{R}$.

Let $D \subset \mathbb{T}^2$ be diffeomorphic to the disc D^2 and denote by $\mathrm{Symp}^D_{c,0}$ the subgroup of Symp_0 consisting of all symplectomorphisms with compact support in D. Then by a similar argument as before, we have again $\mathrm{Symp}^D_{c,0} \subset \mathrm{Ham}(\mathbb{T}^2)$. Now $\mathfrak{Ruelle}$ defines a quasi-morphism on $\mathrm{Symp}_0(\mathbb{T}^2, \omega)$, which is non-trivial on $\mathrm{Symp}^D_{c,0}$, since D is diffeomorphic to D^2 (see Talk 4).

C. Outlook on Calabi-quasimorphisms

In this section, (M, ω) will again denote a closed symplectic manifold. In the last section, we have seen that there exists no non-trivial homomorphism from $\mathrm{Ham}(M)$ to $\mathbb{R}$, but on the other hand, we have managed to construct a non-trivial homogeneous quasi-morphism on $\mathrm{Ham}(\mathbb{T}^2)$ by looking at contractible submanifolds of $\mathbb{T}^2$.

More generally, if U is an orientable 2-dimensional open submanifold of M that satisfies $H^2_c(U, \mathbb{R}) = 0$, then the restriction of any symplectic form ω to U is exact on U. Hence, the Calabi morphism can be defined on the subgroup $\mathrm{Ham}_c(U)$ of $\mathrm{Ham}(M)$ consisting of all Hamiltonian symplectomorphisms with compact support in U. This is in particular the case for submanifolds diffeomorphic to discs - which have the interesting property that the Hamiltonian symplectomorphisms with compact support in U are already all symplectomorphisms with compact support in U - but also for annuli.

Hence, our next aim will be to not only construct a quasi-morphism that is non trivial on all these sub-groups, but which coincides with the respective Calabi morphism of each subgroup. Such a quasi-morphism shall be called a Calabi quasi-morphism. Of course, asking for this property to hold on any 2-dimensional open submanifold of M that satisfies $H^2_c(U, \mathbb{R}) = 0$ would be too restrictive in most cases. Hence, a first possibility is to ask for it to hold only on subspaces diffeomorphic to discs (or annuli). Another approach is to define the so called displacable sets.

Definition 8.16. An open set $U \in M$ is called displacable if ω is exact on U and there exists $f \in \mathrm{Ham}(m)$ such that $f(U) \cap \bar{U} = \emptyset$.

Note finally that neither of these two conditions is stronger than the other one, in general they are incompatible.

Talk 9: Calabi quasimorphisms on Ham T^2

Huaitao, Lukas

A. Definition of Calabi quasimorphisms on $\mathrm{Ham}(\mathbb{T}^2)$

For this section, we again focus on the torus. As the torus is compact, by Theorem 8.14, we know that $\mathrm{Ham}(\mathbb{T}^2,\omega)$ is simple, and hence, there's no nontrivial homomorphism from $\mathrm{Ham}(\mathbb{T}^2,\omega)$ to $\mathbb{R}$. While, there's indeed a nontrivial C^1-continuous homomorphism on $\mathrm{Ham}_c(D^2,\omega)$, namely, the Calabi homomorphism, which is unique up to rescaling (by Lemma 8.5). So we could try to construct a quasimorphism on $\mathrm{Ham}(\mathbb{T}^2,\omega)$, which locally agrees with Calabi homomorphism. We call such a quasimorphism a Calabi quasimorphism. Unfortunately, there does not seem to be a unified definition for "Calabi quasimorphism", so in this talk, in the case of the torus, we define it as follows:

Definition 9.1. Let $\phi : \mathrm{Ham}(\mathbb{T}^2) \to \mathbb{R}$ be a quasimorphism. Suppose that for any disc $D \subset \mathbb{T}^2$, $\phi|_{\mathrm{Ham}_c(D,\omega)}$ is a homomorphism (where $\mathrm{Ham}_c(D,\omega)$ is regarded as a subset of $\mathrm{Ham}(\mathbb{T}^2)$) or equivalently (by Lemma 8.5), suppose for each $D \subset \mathbb{T}^2$, there exists $k \in \mathbb{R}$, such that $\phi|_{\mathrm{Ham}_c(D,\omega)} = k\,\mathrm{CAL}_D$. Then ϕ is called a Calabi quasimorphism.

The goal of this talk is to describe a mechanism to construct a Calabi quasimorphism C_ϕ on $\mathrm{Ham}(\mathbb{T}^2)$ from a homogeneous quasimorphism $\phi : \pi_1(\mathbb{T}^2 \setminus \{0\}) \to \mathbb{R}$, and calculate the values of C_ϕ, when restricted to a specific subgroup $\Gamma \subset \mathrm{Ham}(\mathbb{T}^2)$, with the help of Reeb graph. We follow closely [Py06a].

Throughout this section, we always view $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

B. A general mechanism to construct quasimorphisms on $\mathrm{Symp}_0(\Sigma,\omega)$

Attention: The goal of this subsection is to summarize the idea we adopted in talks 4 and 5 to construct quasimorphisms on $\mathrm{Symp}_0(\Sigma,\omega)$ (or $\mathrm{Symp}_{c,0}(\Sigma,\omega)$ if Σ is not compact), where $\Sigma = D^2, \mathbb{T}^2, \Sigma_g (g \geq 2)$. It means to give a sketch and to describe in general, for that, accuracy is sacrificed. This technique comes from [GG04].

Step 1: For $f \in \mathrm{Symp}_0(\Sigma)$, choose a symplectic isotopy f_t joining id to f.

Step 2: For n distinct points $\{x_1,\cdots,x_n\} \subset \Sigma$, consider the paths $f_t(x_1),\cdots,f_t(x_n)$. We may call $\{x_1,\cdots,x_n\}$ the "starting points".

Step 3: Associate these paths with a "nice" quantity $q(f;x_1,\cdots,x_n)$. A "nice" quantity should be independent of the choice of f_t, and well compatible with path concatenation, i.e.

$$|q(g\circ f;x_1,\cdots,x_n) - q(g;f(x_1),\cdots,f(x_n)) - q(f;x_1,\cdots,x_n)| \leq D,$$

where D is a non-negative number independent of $f,g,x_1,\cdots,x_n$.

Example 9.2. For $n = 1$, we can consider the change of angle of an initial vector along the path, namely $Ang_f(x)$, which finally leads to Ruelle's homogeneous quasimorphism (see Part A of talk 4).

Example 9.3. For $n \geq 2$, fix n distinct points $(x_1^0,\cdots,x_n^0) \in X_n(\Sigma)$, consider the concatenation

$$\gamma(f;x_1,\cdots .x_n) = (((1-t)x_i^0 + tx_i) * (f_t(x_i)) * (tx_i + (1-t)f(x_i)))_{i=1,\cdots,n},$$

which gives us n loops. In knot theory, this is called a pure braid, use $\hat{\gamma}$ to denote its associated link. Then $Sign(\hat{\gamma}(f;x_1,\cdots,x_n))$, where $Sign$ stands for homogenized signature, is such a "good" invariant. This finally gives us linearly independent homogeneous quasimorphisms $\mathfrak{Sign}_{n,D^2}$. (See Part C.1 of talk 4)

Step 4: Integrating to get rid of the starting points. The integration

$$f \mapsto \int_{X_n(\Sigma)} q(f;x_1,\cdots,x_n)dx_1\cdots dx_n,$$

where

$$X_n(\Sigma) := \{(x_1,\cdots,x_n)|x_i \in \Sigma, x_i \neq x_j \text{ when } i \neq j\},$$

turns out to be a quasimorphism (for a typical proof, see Part A of section 4).

Step 5: Homogenization.

Remark 9.4. If $D = 0$ in Step 3, then Step 4 actually defines a homomorphism, Step 5 is then unnecessary. This is exactly the case for $Sign(\hat{\gamma}(f; x_1, x_2))$, thus $\mathfrak{Sign}_{2,D^2}$: $\mathrm{Ham}_c(D^2, \omega) \to \mathbb{R}$ is a homomorphism.

Remark 9.5. When $n \geq 2$, we usually restrict ourselves to paths on D^2 or S^2. Paths on other surfaces are much more complicated. However, for $\Sigma_g, g \geq 2$, by the Poincaré Polygon Theorem, Σ_g can be regared as the quotient manifold of a hyperbolic disc, hence we can lift paths on Σ_g to D^2 and apply the mechanism above. The torus, unfortunately, does not admit a Riemannian metric with constant negative curvature, however, it is indeed a quotient manifold of $\mathbb{R}^2$. So in talk 5, when $n = 1$, we lifted the paths to $\mathbb{R}^2$ and again defined Ruelle's homogeneous quasimorphism. In this section, we focus on the case when $n = 2$, and take advantage of the Lie group structure (inherited from $\mathbb{R}^2$) to modify the mechanism described above to suit the torus.

C. Modified mechanism for $\mathbb{T}^2$

For this subsection, we mean to associate a Calabi quasimorphism C_ϕ defined on $\mathrm{Symp}_0(\mathbb{T}^2, \omega)$ to each homogeneous quasimorphism $\phi : \pi_1(\mathbb{R}^2 - \{0\}) \to \mathbb{R}$. Let a, b be the usual generators of the group $\pi_1(\mathbb{R}^2 - \{0\})$ $(\simeq \mathbb{Z} * \mathbb{Z})$. We wish to prove the following theorem from [Py06a, Theorem 0.1]:

Theorem 9.6. *Suppose $\phi([a, b]) = 1$. Then for any disc $D \subset \mathbb{T}^2$ and any diffeomorphism $f \in \mathrm{Ham}_c(D, \omega)$, one has $C_\phi = 2\,\mathrm{CAL}_D(f)$.*

We first explain the construction of C_ϕ. As $n = 2$, we can use x and y to denote the starting points.

Step 1: For $f \in \mathrm{Symp}_0(\mathbb{T}^2, \omega)$, choose a symplectic isotopy f_t joining id to f.

Step 2 (The trick): View $\mathbb{T}^2$ as the quotient space of $\mathbb{R}^2$, i.e. $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For $(x, y) \in X_2(\mathbb{T}^2)$, consider the paths $f_t(x) - f_t(y)$, which is a path on $\mathbb{T}^2 - \{0\}$ (as $x \neq y, f_t(x) \neq f_t(y)$ for any t).

Step 3: Notice that, there is a natural parameterization of the punctured torus, namely, $\sigma : \mathbb{T}^2 \backslash \{0\} \to ([0, 1) \times [0, 1)) \backslash \{(0, 0)\}$. Fix the base point $x_* = \sigma^{-1}(\frac{1}{2}, \frac{1}{2}) \in \mathbb{T}^2 - \{0\}$. For $v \in \mathbb{T}^2 - \{0\}$, define $\alpha_v(t) := \sigma^{-1}(t\sigma(v) + (1 - t)\sigma(x_*))$, $t \in [0, 1]$, which is a geodesic (under Euclidean metric) going from x_* to v. Define $\alpha(f; x, y)$ to be an element in $\pi_1(\mathbb{T}^2 \backslash \{0\})$, represented by $\alpha_{x-y} * (f_t(x) - f_t(y)) * \bar{\alpha}_{f(x)-f(y)}$. Now, for any quasimorphism $\phi : \pi_1(\mathbb{T}^2 \backslash \{0\}) \to \mathbb{R}$, define $V_f(x, y) := \phi(\alpha(f; x, y))$.

To show that $V_f(x, y)$ is independent of the choice of isotopy, let f'_t be another isotopy joining id to f. Then $f_t * \bar{f}'_t$ forms a loop in $\mathrm{Symp}_0(\mathbb{T}^2, \omega)$. By Lemma 5.3, $f_t * \bar{f}'_t \simeq h_t$, where h_t is a loop consisting of translations, so $f_t \simeq h_t * f'_t$. Notice that $h_t(x) - h_t(y) \equiv x - y$, hence, $f'_t(x) - f'_t(y) \simeq f_t(x) - f_t(y)$, meaning that $\alpha(f; x, y)$ is independent of the choice of isotopy, and naturally, so is $V_f(x, y)$.

To consider the compatibility of $V_f(x, y)$ with path concatenation, for $g \in \mathrm{Symp}_0(\mathbb{T}^2, \omega)$, choose a symplectic isotopy g_t joining id to $g \circ f$, then $f_t * (g_t \circ f)$ is an isotopy joining id to $g \circ f$. Observe that

$$\alpha(g \circ f; x, y) = \alpha(f; x, y)\alpha(g; f(x), f(y)),$$

so we have

$$|\phi(\alpha(g \circ f; x, y)) - \phi(\alpha(f; x, y)) - \phi(\alpha(g; f(x), f(y)))| \leq D_\phi$$

i.e.

$$|V_{g\circ f}(x, y) - V_f(x, y) - V_g(f(x), f(y))| \leq D_\phi.$$

where D_ϕ is the defect of ϕ.

Step 4: We integrate to get rid of the starting points, which gives us a quasimorphism from $\mathrm{Symp}_0(\mathbb{T}^2, \omega)$ to $\mathbb{R}$:

$$f \mapsto \int_{X_2(\mathbb{T}^2)} V_f(x, y)dxdy.$$

Step 5: Homogenization. Define $C_\phi : \mathrm{Symp}_0(\mathbb{T}^2, \omega) \to \mathbb{R}$:

$$C_\phi(f) = \lim_{p\to\infty} \frac{1}{p} \int_{X_2(\mathbb{T}^2)} V_{f^p}(x, y)dxdy = \int_{X_2(\mathbb{T}^2)} \tilde{V}_f(x, y)dxdy$$

where $\tilde{V}_f(x,y) = \lim_{p\to\infty} \frac{1}{p} V_{f^p}(x,y)$.

To prove that C_ϕ is a Calabi quasimorphism, let's first note that the Calabi invariant on $\mathrm{Ham}_c(D)$ can be constructed via the general mechanism in a similar fashion as above.

Back to the general mechanism described in subsection B, assume now $\Sigma = D$, as in Example 9.3, we can define $\gamma(f;x,y)$, a pure braid with two strands, and denote its associated link by $\hat{\gamma}$. We use $n(f;x,y)$ to denote the linking number of $\hat{\gamma}(f;x,y)$, i.e. $n(f;x,y) = lk(\hat{\gamma}(f;x,y))$ (See Definition 4.4).

An equivalent way to define $n(f;x,y)$ is to regard it as the multiplicity of $\gamma(f;x,y)$ with respect to ξ, that is, $\gamma(f;x,y) = n(f;x,y)\xi$, where ξ is the generator of $P_2(D) := \pi_1(X_2(D))$ $(\simeq \mathbb{Z})$, and $lk(\hat{\xi}) = 1$. As $\gamma(g \circ f;x,y) = \gamma(f;x,y)\gamma(g;f(x),f(y))$ (in $P_2(D)$), we have $n(g\circ f;x,y) = n(f;x,y) + n(g;f(x),f(y))$.

Then by Remark 9.4, the integration $\int_{X_2(D)} n(f;x,y)dxdy$ defines a homomorphism on $\mathrm{Ham}_c(D,\omega)$. Due to the uniqueness of homomorphism on $\mathrm{Ham}_c(D)$ (up to rescaling, by Lemma 8.5), there is $k \in \mathbb{R}$, s.t. $\int_{X_2(D)} n(f;x,y)dxdy = k\,\mathrm{CAL}_D(f)$, for any $f \in \mathrm{Ham}_c(D,\omega)$. To calculate the coefficient k, let $\mu : [0,1] \to \mathbb{R}$ be a smooth map that is zero on some neighborhood of 0 and 1 (in Section 4, we use the notation ω, here to avoid the confusion with the symplectic form, we use μ). Define the symplectic diffeomorphism F_μ on the disc (in polar coordinates) by $F_\mu(R,\theta) = (R, \theta + 2\pi\mu(R))$. Then apply Lemma 4.11 and Lemma 8.2, to calculate $\mathfrak{Sign}_{2,D}(F_\mu) = -4\,\mathrm{CAL}_D(F_\mu)$. A fact in knot theory is that $-2n(f;x,y) = Sign(\hat{\gamma}(f;x,y))$, so $\int_{X_2(D)} n(F_\mu;x,y) = 2\,\mathrm{CAL}(F_\mu)$, meaning $k=2$.

Now, it suffices to check for any $f \in \mathrm{Symp}_{c,0}(D)$, $C_\phi(f) = \int_{X_2(D)} n(f;x,y)dxdy$, or equivalently, $\int_{X_2(\mathbb{T}^2)} \tilde{V}_f(x,y)dxdy = \int_{X_2(D)} n(f;x,y)dxdy$. Let's divide the integration on the LHS into 3 parts, and consider separately.

Case 1: $x, y \notin D$.

Then $f_t(x) \equiv x$ and $f_t(y) \equiv y$, therefore, $\alpha(f^p;x,y) = e$, where e is the identity of $\pi_1(\mathbb{T}^2 - \{0\})$. So,

$$\tilde{V}_f(x,y) = \lim_{p\to\infty} \frac{1}{p} V_{f^p}(x,y) = \lim_{p\to\infty} \frac{1}{p}\phi(e) = 0.$$

Case 2: $x \in D, y \notin D$ or $x \notin D, y \in D$.

Let's assume that $x \in D, y \notin D$, then

$$\alpha(f;x,y) = \alpha_{x-y} * (f_t(x) - f_t(y)) * \bar{\alpha}_{f(x)-f(y)} = \alpha_{x-y} * (f_t(x) - y) * \bar{\alpha}_{f(x)-f(y)}.$$

Consider the lift of $\alpha(f;x,y)$ in the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$. As $f_t(x) - y$ is always in a disc, the lift of it is also within a disc of $\mathbb{R}^2\setminus\mathbb{Z}^2$, which is contractible, so the lift of $f_t(x) - y$ is homotopic to a straight line with length less than two times of the radius of the disc (all under Euclidean metric). Hence, the lift of $\alpha(f;x.y)$ can always be represented by concatenation of three straight lines with length less than 1, which means there are only finite possibilities. Actually, a more careful treatment tells us $\alpha(f;x,y) \in \{e, a, b, a^{-1}, b^{-1}\}$, where a, b are the generators of $\pi_1(\mathbb{T}^2\setminus\{0\})$. Let $M := \max\{|\phi(e)|, |\phi(a)|, |\phi(b)|, |\phi(a^{-1})|, |\phi(b^{-1})|\}$, then

$$|\tilde{V}_f(x,y)| = \lim_{p\to\infty} \frac{1}{p} |V_{f^p}(x,y)| \le \lim_{p\to\infty} \frac{1}{p} M = 0.$$

Case 3: $x, y \in D$.

Consider the map:

$$\begin{aligned} u : X_2(D) \to & X_2(\mathbb{T}^2) \to \mathbb{T}^2\setminus\{0\} \\ (x,y) \mapsto & (x,y) \mapsto x - y \end{aligned}$$

then it induces a map $u_* : \pi_1(X_2(D), (x_0, y_0)) \to \pi_1(\mathbb{T}^2\setminus\{0\}, x_0 - y_0)$, where

$$\begin{aligned} \gamma(f;x,y) = & ((1-t)x^0 + tx, (1-t)y^0 + ty) * (f_t(x), f_t(y)) * (tx^0 + (1-t)f(x), ty^0 + (1-tf(y))) \\ & \mapsto ((1-t)(x^0 - y^0) + t(x-y)) * (f_t(x) - f_t(y)) * (t(x^0 - y^0) + (1-t)(f(x) - f(y))). \end{aligned}$$

Let $\beta_{(x,y)} := ((1-t)x^0+tx, (1-t)y^0+ty)_{t\in[0,1]}$. Then calculate (the reader is encouraged to assume $x_* = x^0 - y^0$ to avoid the tedious calculation and grab the main idea)

$$\begin{aligned}\alpha(f;x,y) =&\alpha_{x-y} * (f_t(x) - f_t(y)) * \bar{\alpha}_{f(x)-f(y)}\\ =&\alpha_{x-y} * \overline{u(\beta_{(x,y)})} * u(\beta_{(x,y)}) * (f_t(x) - f_t(y)) * \overline{u(\beta_{(f(x),f(y))})}\\ &* u(\beta_{(f(x),f(y))}) * \bar{\alpha}_{f(x)-f(y)}\\ =&\alpha_{x-y} * \overline{u(\beta_{(x,y)})} * u_*(\gamma(f;x,y)) * u(\beta_{(f(x),f(y))}) * \bar{\alpha}_{f(x)-f(y)}\\ =&\alpha_{x-y} * \overline{u(\beta_{(x,y)})} * \bar{\alpha}_{x^0-y^0} * \alpha_{x^0-y^0} * u_*(\gamma(f;x,y)) * \bar{\alpha}_{x^0-y^0} * \alpha_{x^0-y^0}\\ &* u(\beta_{(f(x),f(y))}) * \bar{\alpha}_{f(x)-f(y)}\end{aligned}$$

Again, by a similar argument as in Case 2, $\phi(\alpha_{x-y} * \overline{u(\beta_{(x,y)})} * \bar{\alpha}_{x^0-y^0})$ and $\phi(\alpha_{x^0-y^0} * u(\beta_{(f(x),f(y))}) * \bar{\alpha}_{f(x)-f(y)})$ have only finite many possible values, and therefore have an upper bound M independent of x, y, f. So,

$$V_f(x,y) = \phi(\alpha(f;x,y)) \leq 2D_\phi + n(f;x,y)\phi(u_*(\xi)) + 2M.$$

Observe that $u_*(\xi)$ is conjugate to $[a,b]$, as ϕ is homogenous, $\phi(u_*(\xi)) = \phi([a,b]) = 1$. Hence,

$$\Big|\int_{X_2(D)} V_f(x,y)dxdy - \int_{X_2(D)} n(f;x,y)dxdy\Big| \leq 2(D_\phi + M)area(D)^2.$$

Notice that $\int_{X_2(D)} V_f(x,y)dxdy$ also defines a quasimorphism on $\mathrm{Ham}_c(D)$, and $\int_{X_2(D)} \tilde{V}_f(x,y)dxdy$ is its homogenization. By uniqueness of homogeneous representative of a quasimorphism(See Proposition 1.5), $\int_{X_2(D)} \tilde{V}_f(x,y)dxdy = \int_{X_2(D)} n(f;x,y)dxdy$, which completes the proof.

D. Interlude: Morse Functions

We will now look at an interesting subgroup of $\mathrm{Ham}(\mathbb{T}^2,\omega)$, but in order to describe it we need some prerequisites.

Definition 9.7. A function $F \in C^\infty(M)$ on a smooth manifold M is called a Morse function if its critical points are non degenerate.

For us the most important fact about the critical points is that they are isolated and since we are considering $\mathbb{T}^2$ here, which is compact, a Morse function will have only finitely many critical points. However we want some more structure on the critical points.

Lemma 9.8. *For every smooth manifold M there exists a Morse function F such that for any distinct critical points $x, y \in M$ we have $F(x) \neq F(y)$.*

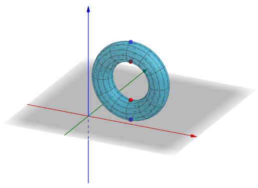
Now we will fix such a function F for the torus, with critical points $x_1, ..., x_n$ and critical values $\lambda_i = F(x_i)$ such that $\lambda_1 < ... < \lambda_n$.

A convenient way of picturing a Morse function is as the height function of a particular embedding of the torus in $\mathbb{R}^3$, see figure 10, where critical points are either valleys (local minima), peaks (local maxima) or saddles.

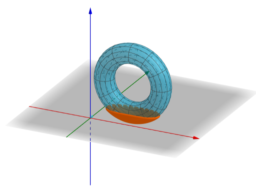
Now we move on to defining our subgroup.

Definition 9.9. $\mathcal{F} := \{H \in C^\infty(\mathbb{T}^2) | \omega(X_H, X_F) = 0\}$ where X_H is the vector field such that $\iota_{X_H}\omega = dH$.
We also define the subgroup of flows at time $t = 1$ of these vector fields $\Gamma := \{\varphi_H^1 | H \in \mathcal{F}\} \leq Ham(\mathbb{T}^2, \omega)$

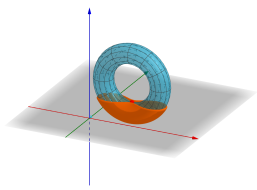
As ω is non-degenerate, we have $\forall H \in \mathcal{F}\ \exists\lambda : \mathbb{T}^2 \to \mathbb{R} : X_H = \lambda X_F$, so all the corresponding vector fields are parallel and we obtain that $\forall H_1, H_2 \in \mathcal{F} : \varphi_{H_1}^1 \circ \varphi_{H_2}^1 = \varphi_{H_2}^1 \circ \varphi_{H_1}^1$, so Γ is an abelian subgroup of $\mathrm{Ham}(\mathbb{T}^2,\omega)$.



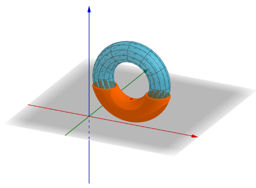
(A) Critical points of the height function



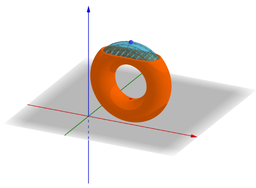
(B) F after the first critical point



(C) F at the second critical point



(D) F between the 2nd and 3rd critical points



(E) F between the last critical points

FIGURE 10. F as a height function

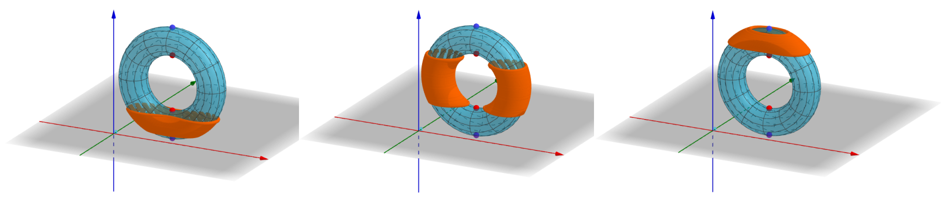
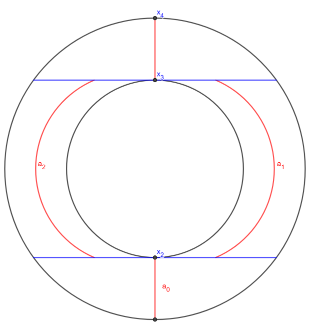
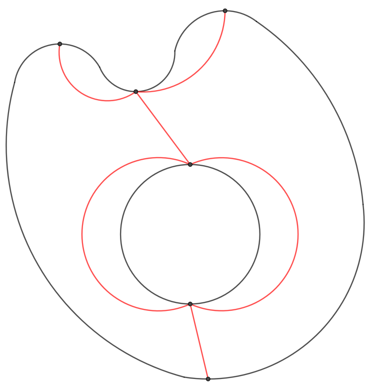


FIGURE 11. Cylinders



(A) Side view of construction of the Reeb Graph



(B) Example with a tree

FIGURE 12. Reeb Graphs

E. Interlude: The Reeb Graph

In order to define the Reeb graph we will denote by K_i the connected component of $F^{-1}(x_i)$ containing x_i (this is either a single point or a figure 8) and note that $\mathbb{T}^2 \setminus \cup_{i=1}^{n} K_i$ is a collection of disjointed cylinders, see figure 11.

We now construct the Reeb (multi-)graph $\mathcal{G}$: To each $K_i/x_i/\lambda_i$ we associate a vertex s, and to each cylinder we associate an edge a connecting s_i to s_j $(\lambda_i < \lambda_j)$ from the respective K_i that bound it. We can consider a as the interval $]\lambda_i, \lambda_j[$. See figure 12.

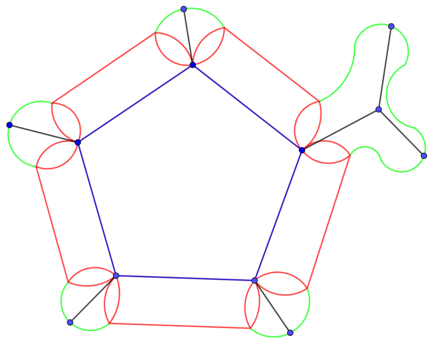


FIGURE 13. Cycle in blue, the corresponding cylinders in red, trees in black with corresponding cylinders in green

As the torus has genus 1, $\mathcal{G}$ has a unique cycle $\mathcal{G}'$ of vertices $s_1, ..., s_k$ with edges $a_1, ..., a_k$ where a_i connects s_i to s_{i+1} (with the convention that $s_{k+1} = s_1$). To see that $\mathcal{G}'$ must exist, one can picture the torus as a "loop" of cylinders, which correspond to the a_i. Uniqueness follows from a similar argument showing that a second loop would increase the genus. The cylinders corresponding to the edges of $\mathcal{G}'$ cannot attach to each other in order to "close" the surface, so at each vertex we need to "cap off" the torus and we get trees $T_1, ..., T_k$ such that T_i intersects $\mathcal{G}'$ only at the vertex s_i. See figure 13.

In other words $\mathcal{G} = \mathcal{G}' \cup_{i=1}^k T_i$.

F. Expressing C_ϕ on Γ

Now we can define a projection $p : \mathbb{T}^2 \to \mathcal{G}$ such that for any $x \in a =]\lambda_i, \lambda_j[$ we have $p^{-1}(x)$ is the connected component of $F^{-1}(x)$ in the cylinder corresponding to a.

As any $H \in \mathcal{F}$ shares level sets with F we now can find a functions $H_\mathcal{G} : \mathcal{G} \to \mathbb{R}$ such that $H = H_\mathcal{G} \circ p$. We can now state the theorem ([Py06a, Theorem 0.2]) that we want to show.

Theorem 9.10. *For any $\phi : \pi_1(\mathbb{T}^2 \setminus \{0\}) \to \mathbb{R}$ and any $H \in \mathcal{F}$ we have*

$$C_\phi(\varphi_H^1) = 2\sum_{i=1}^{k} \int_{p^{-1}(T_i)} (H - H_\mathcal{G}(s_i))\omega$$

Particularly interesting is that the right hand side of this equation is completely independent of ϕ, with all quantities being defined by our Morse function F and H.

G. Sketch of Proof

For the proof we need to introduce some more notation: For an edge a we denote by a^+, a^- the vertices it borders such that $F_\mathcal{G}(a^+) > F_\mathcal{G}(a^-)$, denote these values by t_a^+ and t_a^- respectively. Up to changing the order of $\mathcal{G}'$ we may assume $a_1^- = s_1$ and $a_1^+ = s_2$. We can also parametrize $p^{-1}(a)$ as $(t, \theta) \in]t_a^-, t_a^+[\times \mathbb{R}/\mathbb{Z}$ such that $\omega = d\theta \wedge dt$.

With this parametrization we have $X_F(t, \theta) = \nu_F(t)\frac{\partial}{\partial\theta}$ where $\nu_F > 0$. And for $H \in \mathcal{F}$ we also have $X_H(t, \theta) = \nu_H(t)\frac{\partial}{\partial\theta}$.

Note that we now have $\int_{t_a^-}^{t_a^+} \nu_h(t)dt = H_\mathcal{G}(a^+) - H_\mathcal{G}(a^-)$.

Finally for an edge a, let $\gamma(u)$ be a loop in $p^{-1}(a)$ along $\frac{\partial}{\partial\theta}$ and q a point outside of $p^{-1}(a)$, so we can define $c_a = [\gamma - p]$ in $\pi_1(\mathbb{T}^2 \setminus \{0\})$.

Lemma 9.11. *For $H \in \mathcal{F}$ we have $\sum_{i=1}^k (H_\mathcal{G}(a_i^+) - H_\mathcal{G}(a_i^-))\phi(c_{a_i}) = 0$*

Proof. To see this we will turn the sum into a telescoping sum where $s_{k+1} = s_1$ means it is 0.

We consider two cases for a_i: **Case 1: $a_i^+ = s_{i+1}$.** In this case a_i is oriented the same way as a_1 and thus we get that $c_{a_i} = c_{a_1}$. **Case 2: $a_i^+ = s_i$.** In this case we note that by "sliding" the γ_1 used in the definition for c_{a_1} along the loop of the torus (represented by $\mathcal{G}'$

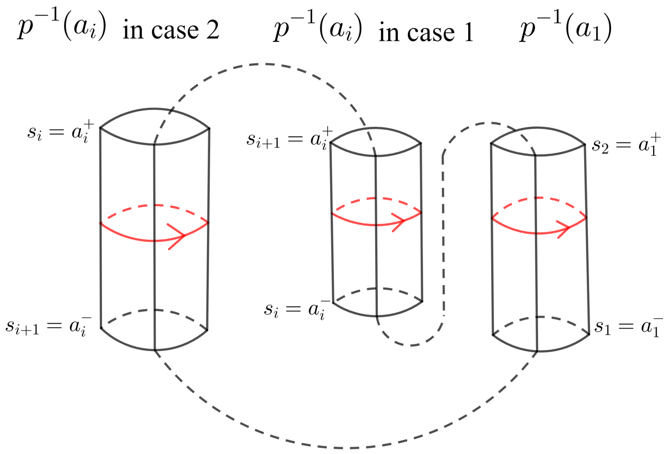


FIGURE 14. The loop in red is the respective γ for that edge

in the graph) it has the reverse orientation as the γ_i used for c_{a_i}. So we have $c_{a_i} = -c_{a_1}$. For both these cases see figure 14.

As ϕ is homogeneous we get for any a_i:

$$(H_{\mathcal{G}}(a_i^+) - H_{\mathcal{G}}(a_i^-))\phi(c_{a_i}) = (H_{\mathcal{G}}(s_{i+1}) - H_{\mathcal{G}}(s_i))\phi(c_{a_1})$$

□

We now can start proving the theorem by calculating $C_\phi(\varphi_H^1) = \int_{X_2(\mathbb{T}^2)} \tilde{V}_{\varphi_H^1}(x,y)dxdy$. We can split this into the parts $p^{-1}(a) \times p^{-1}(b)$ for edges a and b. (Note that the preimage of the vertices has area 0.) We further distinguish 3 cases.

(1) a, b are both edges in $\mathcal{G}'$.
(2) One is in $\mathcal{G}'$ the other in a T_i.
(3) Both are in T_i's.

Case 1 Let $x \in p^{-1}(a)$ and $y \in p^{-1}(b)$. Note that we may assume that $p(x) \neq p(y)$. Now we can note that the paths $\varphi_H^t(x)$ and $\varphi_H^t(y)$ simply move in their level sets in a along $\frac{\partial}{\partial\theta}$ at the rate $\nu_H(x)$ and $\nu_H(y)$ respectively and that these paths do not intersect. This allows us to write:

$$(\varphi_H^t(x) - \varphi_H^t(y))_{t\in[0,n]} \simeq (\varphi_H^t(x) - y)_{t\in[0,n]} * (\varphi_H^n(x) - \varphi_H^t(y))_{t\in[0,n]}$$

Letting s_* be the automorphism on $\pi_1(\mathbb{T}^2 \setminus \{0\})$ induced by $u \mapsto -u$ we get:

$$\tilde{V}_{\varphi_H^1}(x,y) = \nu_H(x)\phi(c_a) + \nu_H(y)\phi(s_*(c_b))$$

Integrating over $p^{-1}(a) \times p^{-1}(b)$ we get:

$$\int_{p^{-1}(a)\times p^{-1}(b)} \tilde{V}_{\varphi_H^1}(x,y)dxdy = (H_{\mathcal{G}}(a^+) - H_{\mathcal{G}}(a^-))\phi(c_a)area(p^{-1}(b)) + (H_{\mathcal{G}}(b^+) - H_{\mathcal{G}}(b^-))\phi \circ s_*(c_b)area(p^{-1}(a))$$

Thus with the lemma summing over a and b in $\mathcal{G}'$ gives us 0.

Case 2 As the subcases are symmetrical we will only consider $a \in \mathcal{G}'$ and $b \in T_i$. Further, since the intersection of the tree and the cycle is s_i, we can homotop the path $\varphi_H^t(b)$ around $p^{-1}(T_i)$ and contract it to a point, see figure 15, and similar to the last case we have:

$$(\varphi_H^t(x) - \varphi_H^t(y))_{t\in[0,n]} \simeq (\varphi_H^t(x) - y)_{t\in[0,n]}.$$

Integrating and summing then gives us 0 again with the lemma.

Case 3 If a and b are in different trees we note that, as in case 2, we can homotop their respective paths to constant paths as there is no obstruction and get $\tilde{V}_{\varphi_H^1}(x,y) = 0$, so we only need to consider $a = b$ in the tree T_i.

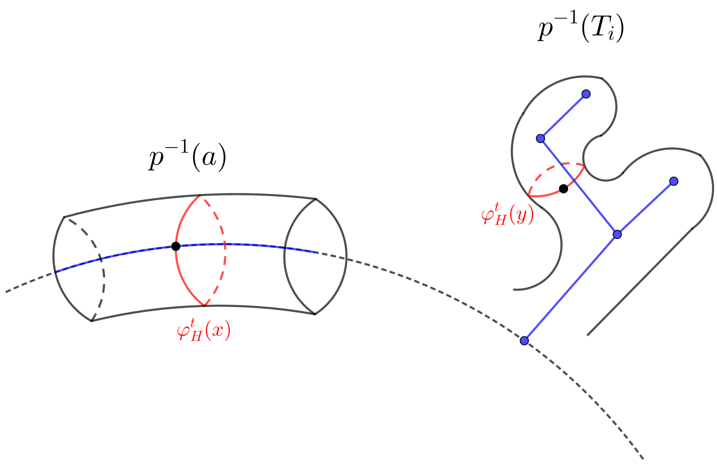


FIGURE 15. Case 2

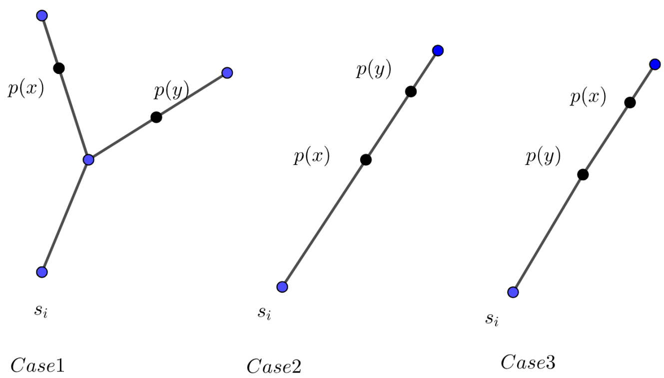


FIGURE 16

Now we have again three cases, see figure 16. In the first we can argue as above that $\tilde{V}_{\varphi^1_H}(x, y) = 0$. The last 2 are symmetric, so we only have to consider the second one.
We will need to introduce some new notation needed for the calculation. Let $\epsilon(a)$ be equal to 1 if F grows as we approach s_i (visually this means the tree is "below" s_i) and -1 if F decreases as it approaches s_i (the tree is "above" s_i). See figure 17 for the case that $\epsilon(a) = -1$.

As before we homotop the path $\varphi^t_H(y)$ to the constant path at y and then the path $\varphi^t_H(x)$ circles this point at the rate $\nu_H(x)$, in the positive direction if $\epsilon(a) = -1$ and in the negative direction if $\epsilon(a) = 1$. Thus we have $\tilde{V}_{\varphi^1_H}(x, y) = -\epsilon(a)\nu_H(x)$.

For a point $u \in T_i$ that is not a vertex, let $D(u)$ be the connected component of $p^{-1}(T_i \setminus \{u\})$ on which F increases, let $\chi(u) = area(D(u))$ and $\chi(a^+) = \lim_{u\in a, u\to a^+} \chi(u)$ and $\chi(a^-) = \lim_{u\in a, u\to a^-} \chi(u)$, note that these latter two may depend on a, not just the vertices if a is at a fork of T_i, see figure 18.

Thanks to our parametrization $\omega = d\theta \wedge dt$ we have that $\frac{\partial}{\partial t} area(D(p(t, \theta))) = \epsilon(a)$.

A very tedious integration by parts gives us the following:

$$\int_{x\in p^{-1}(a), y\in D(p(x))} \tilde{V}_{\varphi^1_H}(x, y)dxdy = \int_{p^{-1}(a)} H\omega - \epsilon(a)(H_{\mathcal{G}}(a^+)\chi(a^+) - H_{\mathcal{G}}(a^-)\chi(a^-)).$$

The only $\epsilon(a)H_{\mathcal{G}}(a^{+-})\chi(a^{+-})$ terms that do not cancel out when summing over $a \in T_i$, are the leaves of T_i, of which only s_i as $\chi(s_i) = area(T_i)$ and the others are 0, and it

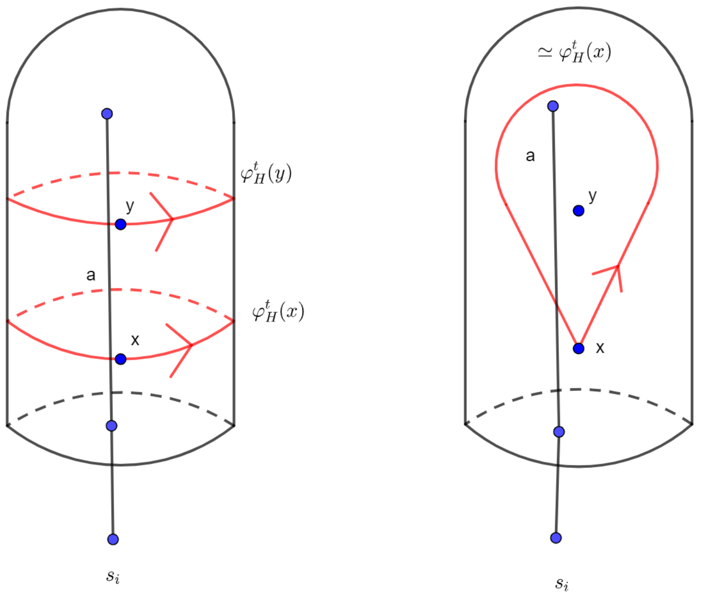


FIGURE 17. If $\epsilon(a) = -1$

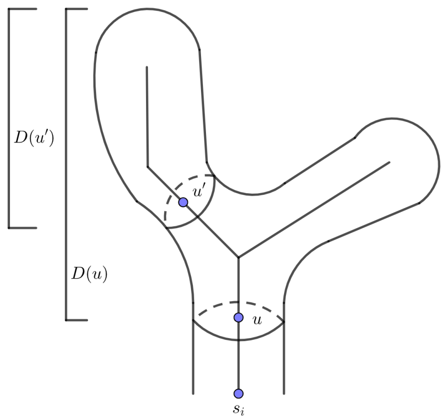


FIGURE 18. $D(u)$ above and below a fork

appears with the same sign as $\epsilon(a)$. So we have:

$$\begin{aligned}\int_{x\in p^{-1}(T_i),y\in D(p(x))} \tilde{V}_{\varphi_H^1}(x,y)dxdy &= \int_{p^{-1}(a)} H\omega - H_{\mathcal{G}}(s_i)area(T_i)\\ &= \int_{p^{-1}(T_i)} (H - H_{\mathcal{G}}(s_i))\omega\end{aligned}$$

Symmetry then gives us:

$$\int_{x,y\in p^{-1}(T_i)} \tilde{V}_{\varphi_H^1}(x,y)dxdy = 2\int_{p^{-1}(T_i)} (H - H_{\mathcal{G}}(s_i))\omega$$

And summing over the T_i's gives us the theorem.

Talk 10: Calabi quasimorphisms on $\mathrm{Ham}(\Sigma_g), g \geq 2$

Jiahui, Konstantin

A. Main theorem and some notations

Theorem 10.1. *Let (S, ω) be a closed oriented symplectic surface of genus at least 2. There exists homogeneous quasimorphsim*

$$\mathfrak{Cal}_S : \mathrm{Ham}(S, \omega) \to \mathbb{R}$$

which is invariant under $\mathrm{Symp}(S, \omega)$ *conjugation, and equal to the Calabi homomorphism for diffeomorphisms supported on a disc or an annulus.*

Recall by Moser's theorem, any two volume forms of equal total volume can be transformed into each other via some diffeomorphism isotopic to the identity, so the choice of ω does not matter. Assume ω is the induced area form by some metric of constant curvature on S.

Let $\widetilde{S}$ be the universal cover of S with induced metric. Let $M = \mathrm{UT}(S), \widetilde{M} = \mathrm{UT}(\widetilde{S})$ be the unit tangent bundles. A point $(\widetilde{x}, v) \in \widetilde{M}$, where $\widetilde{x} \in \widetilde{S}$ and $v \in \mathrm{UT}_{\widetilde{x}}(\widetilde{S})$, identifies a geodesic ray on $\widetilde{S}$ starting at $\widetilde{x}$ pointing at direction v, which gives a point in the boundary at infinity $p_\infty(\widetilde{x}, v) \in S^1_\infty$. Given a path $\gamma : [0,1] \to \widetilde{M}$, p_∞ sends it to a path on $S^1_\infty \cong \mathbb{R}/\mathbb{Z}$ and we can lift it to $\widetilde{\gamma} : [0,1] \to \mathbb{R}$. Define the index of γ to be

$$n(\gamma) = \lfloor \widetilde{\gamma}(1) - \widetilde{\gamma}(0) \rfloor$$

which records the "number of turns" that γ does on the S^1_∞. Note that this generalizes Definition 5.8.

Let X be the vector field on M induced by the S^1-action on each fiber. We consider an 1-form α on M, such that $\alpha(X) = 1$ and $\iota_X(d\alpha) = 0$. Such an α exists, and can be constructed using partition of unity, where locally it is $xdy + d\theta$.

B. Construction of $\mathfrak{Cal}_S$

Let (f_t) be a Hamiltonian isotopy on S with generating vector field X_t and Hamiltonian functions H_t. Let $\widehat{X_t}$ be the horizontal lift of X_t to TM that satisfies $\alpha(\widehat{X_t}) = 0$. Set

$$\phi(X_t) = \widehat{X_t} - (H_t \circ \pi) X$$

Let $(\Phi(f_t))$ be the flow along $\phi(X_t)$ on M, and lift to an isotopy (F_t) on $\widetilde{M}$.

For $\widetilde{x} \in \widetilde{S}$, define

$$\widetilde{Ang}_{f_1}(\widetilde{x}) = -\inf\{n(F_t(\widetilde{x}, v)) : v \in \mathrm{UT}_{\widetilde{x}}(\widetilde{S})\}$$

Arguments from talk 5 (see (5.10)-(5.13)) shows that this infimum exists, is invariant under deck transformation, therefore defines a map $Ang_f(x) = \widetilde{Ang}_f(\widetilde{x})$, such that

$$|Ang_{fg}(x) - Ang_g(x) - Ang_f(g(x))| \leq 4$$

Integrating gives us a quasimorphism, as we have seen many times:

$$\mathrm{CAL}_S(f) = \int_S Ang_f \omega$$

Homogenize and get

Definition 10.2.

$$\mathfrak{Cal}_S(f) = \lim_{p \to \infty} \frac{1}{p} \int_S Ang_{f^p} \omega$$

Remark 10.3. In talk 5, instead of F_t above, we used the map $(\widetilde{x}, v) \mapsto (\widetilde{f}_t, d\widetilde{f}_t(\widetilde{x})(v))$. The same arguments still work. The reason for using F_t is so that the resulting quasimorphism satisfies the extra conditions (see [Py06a] (2.1) for more motivation). The index n is independent of the choice of metric. The quasimorphism is independent of the choice of the metric, of H_t, and of α.

Since conjugating is the same as computing in the pullback metric (see Prop 6.26), $\mathfrak{Cal}_S$ is invariant under conjugation by symplectomorphisms. Also, we can change H_t by a constant so that $\int_S H_t \omega = 0$.

C. Restriction to diffeomorphisms supported on U homeomorphic to disc

Let $U \subseteq S$ be homeomorphic to a disc on which $\omega = d\lambda$. Recall from Lemma 8.2, for $G(x) = \int_0^1 (H_s + \lambda(X_s)) f_s(x) ds$, we have

$$\mathfrak{Cal}_U(f) = \int_U \int_0^1 H_t dt \omega = \frac{1}{2} \int_U G\omega$$

By subadditive ergodic theorem, the functions $\frac{1}{p} Ang_{f^p}$ converge ω-almost everywhere to some measurable function $\widehat{Ang}_f$, so

$$\mathfrak{Cal}_S(f) = \int_S \widehat{Ang}_f \omega$$

Proposition 10.4.

$$\widehat{Ang}_f(x) = \lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} G(f^k(x))$$

Proof. Let $\pi : M \to S$ be the bundle projection. Pick a trivialization $U \times S^1 \to \pi^{-1}(U) = \mathrm{UT}(U)$. We shall write points of $\mathrm{UT}(U)$ in form (x, θ). Then $X = \frac{\partial}{\partial\theta}$ and we can take α to be $d\theta + \pi^*\lambda$.

Let $f_{p,t}$ be the Hamiltonian isotopy $(f_t) * (f_t \circ f) * \cdots * (f_t \circ f^{p-1})$, starting at id and ending at f^p. Let $X_{p,t}$ be its generating vector field and $H_{p,t}$ be the Hamiltonian with $\int_S H_{p,t}\omega = 0$. Note that on the k-th segment, $X_{p,t} = (f^{k-1})^* X_t, H_{p,t} = (f^{k-1})^* H_t$.

Recall $\Phi(f_{p,t})$ is the flow of $\phi(X_{p,t}) = \widehat{X_{p,t}} - (H_{p,t} \circ \pi)\frac{\partial}{\partial\theta}$. Since $\alpha(\widehat{X_{p,t}}) = 0$ by definition, we have $0 = d\theta(\widehat{X_{p,t}}) + \pi^*\lambda\widehat{X_{p,t}}$. Therefore for $\theta \in S^1$ and $x \in U$,

$$\begin{aligned}\Phi(f_{p,t})(x,\theta) &= \left(f_{p,t}(x), \theta + \int_0^t (\phi(X_{p,s})d\theta)(f_{p,s}(x))ds \right) \\ &= \left(f_{p,t}(x), \theta - \int_0^t (\lambda X_{p,s} + H_{p,s})(f_{p,s}(x))ds \right)\end{aligned} \tag{38}$$

Let $v(t), v_1(t), v_2(t)$ denote the following paths:

$$\Phi(f_{p,t})(x,\theta), \quad (f_{p,t}(x), \theta)$$

$$\left(f^p(x), \int_0^t (\lambda X_{p,s} + H_{p,s})(f_{p,s}(x))ds \right)$$

Let $\approx$ denote that two elements differ by a bounded quantity (independent of p), then

$$Ang_{f^p}(x) \approx -n(\widetilde{v}) \approx -n(\widetilde{v_1} * \widetilde{v_2}) \approx -n(\widetilde{v_1}) - n(\widetilde{v_2})$$

Let K be a compact set that $\mathrm{supp}(f) \subseteq K$. Fix a point $x_0 \in U$. For $x \in K$, let $\alpha_{x_0 x}$ be a path from x_0 to x. By compactness, $n(\widetilde{\alpha_{x_0 x}})$ is bounded. Let γ_{x,f^p} be the loop $\alpha_{x_0 x} * (f_{p,t}(x)) * \overline{\alpha_{x_0, f^p(x)}}$. Since U is simply connected, it lifts to a contractible loop $\widetilde{\gamma_{x,f^p}, \theta}$. Therefore

$$-n(\widetilde{v_1}) \approx -n(\widetilde{\gamma_{x,f^p}, \theta}) - n(\widetilde{\alpha_{x_0 x}}) - n(\widetilde{\alpha_{x_0, f^p(x)}}) \approx 0$$

On the other hand,

$$-n(\widetilde{v_2}) \approx \int_0^1 (\lambda X_{p,s} + H_{p,s})(h_{p,s}(x))ds = \sum_{k=0}^{p-1} \int_0^1 (\lambda X_s + H_s) f_s(f^k(x))ds = \sum_{k=0}^{p-1} G(f^k(x))$$

$$\widehat{Ang}_f(x) = \lim_{p\to\infty} \frac{1}{p} Ang_{f^p}(x) = \lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} G(f^k(x))$$

□

Now it follows from the Birkhoff Ergodic theorem, $\int_U \widehat{Ang}(x,f)\omega = \int_U G\omega$.

An important difference from talk 5 is that when $x \notin U$, although f fixes x, $Ang(x, f)$ may be non-zero. In fact, in this case $X_t = 0$ and H_t is some constant which might not be zero. We have $\phi(X_t) = H_t \circ \pi \frac{\partial}{\partial \theta}$, so

$$Ang(x, f^p) \approx \sum_0^{p-1} \int_0^1 H_t(f_t(f^k(x)))dt = p \int_0^1 H_t dt$$

$$\widehat{Ang}_f(x) = \int_0^1 H_t dt$$

Finally, we conclude that $\mathfrak{Cal}_S(f) = \mathfrak{Cal}_U(f)$ when (f_t) is supported on U, as follows:

$$\begin{aligned}
\mathfrak{Cal}_S(f) &= \int_S \widehat{Ang}_f \omega \\
&= \int_{S-U} \widehat{Ang}_f \omega + \int_U \widehat{Ang}_f \omega \\
&= \int_{S-U} \int_0^1 H_t dt \omega + \int_U G\omega \\
&= \int_{S-U} \int_0^1 H_t dt \omega + 2 \int_U \int_0^1 H_t dt \omega \\
&= \int_U \int_0^1 H_t dt \omega = \mathfrak{Cal}_U(f)
\end{aligned} \tag{39}$$

Remark 10.5. During the proof we used the path $\widetilde{v_1}$, which in our simply connected case gives $n(\widetilde{(v_1)}) = 0$ by contractibility. In general, given a loop γ on any connected open set U and a trivialization $U \times S^1 \to \pi^{-1}(U)$, we can define $\phi(\gamma) = n(\gamma(t), z_0)$ where $z_0 \in S^1$ is fixed. Then $\phi : \pi_1(U) \to \mathbb{R}$ is a quasimorphism independent of the choice of z_0, the basepoint of the loop, nor the metric. We shall see more use of this in the following sections.

D. Reminder of the Reeb graph

Recall the definition of a Morse function definition 9.7, which is a smooth function $F \in C^\infty(S)$ such that all critical points are non-degenerate. So let $F : S \to \mathbb{R}$ be a Morse function, for which all critical values are distinct. Denote by $x_1, \ldots, x_l$ the critical points with corresponding critical values $\lambda_j = F(x_j), j \in \{1, \ldots, l\}$, such that $\lambda_1 < \cdots < \lambda_l$.

Now consider the space

$$\mathcal{F} := \{H : S \to \mathbb{R} \mid H \in C^\infty,\ \omega(X_H, X_F) = 0\}$$

of functions on S which commute with F (X_G denotes the symplectic gradient of a function G). This gives rise to the set

$$\Gamma := \{\phi_H^1 \mid H \in \mathcal{F}\}$$

which is an abelian subgroup of $Ham(S, \omega)$ where ϕ_H^t denotes the flow of X_H. Therefore the restriction of $\mathfrak{Cal}_S$ is a homomorphism on Γ, which we will calculate in the following section explicitly. However, first we have to look at the Reeb graph $\mathcal{G}$ of our Morse function F. (For the construction of the Reeb graph see section E.) We have a natural map $p_{\mathcal{G}} : S \to \mathcal{G}$ such that for every $H \in \mathcal{F}$ we can write $H = H_{\mathcal{G}} \circ p_{\mathcal{G}}$, where $H_{\mathcal{G}}$ is defined on $\mathcal{G}$. The reason for this is that all $H \in \mathcal{G}$ share the level sets with F.

Now we will prune our graph $\mathcal{G}$ to obtain another graph $\mathcal{G}'$ as follows. One can check that $\mathcal{G}$ has only vertices of degree 1 or 3. For all vertices v of degree 1 we will contract v along its incident edge until it coincides with the vertex adjacent to it and we denote the graph obtained in that way $\mathcal{G}'$. We do this iteratively until there are no more vertices of degree 1 left. Hence $\mathcal{G}'$ will only consist of vertices with degree 2 or 3. The number of vertices with degree 3 will be $2g - 2$. Indeed, the quantity $\sum_v 2 - \deg(v)$ is equal to the Euler characteristic of the surface, which is $2 - 2g$ and stays invariant under our vertex contraction. Observe that if we contract a leave of our graph, we loose one vertex, but also the unique edge incident with it, which hence amounts to no change in the identity above. Finally denote with $\mathcal{V}$ the set of all the $(2g-2)$ vertices with degree 3 in the graph $\mathcal{G}'$.

E. Some concrete calculations of the Calabi quasimorphism

Our goal in this section is to prove the following theorem. In the following theorem we assume that our 2-form ω has total area of $(2g-2)$.

Theorem 10.6. *If H is in $\mathcal{F}$, we have*

$$\mathfrak{Cal}_S(\varphi_H^1) = \int_S H\omega - \sum_{v\in\mathcal{V}} H_{\mathcal{G}}(v).$$

Denote by U the open set $S-\{x_l\}$ and fix a trivialisation of the fiber bundle $\pi : M \to S$ over U. Like in the previous section this yields a primitive element λ of ω in U and a homogeneous quasimorphisms ϕ on $\pi_1(U)$ like described above. For an arc a of the Reeb graph $\mathcal{G}$ we denote by a^+ and a^- the nodes incident to a with the convention that $F_{\mathcal{G}}(a^-) < F_{\mathcal{G}}(a^+)$. By construction, we can parametrize the preimage of an arc, $p_{\mathcal{G}}^-1(a)$, by $(\theta, t) \in \mathbb{R}/\mathbb{Z} \times (t_-, t_+)$, such that we can write $\omega = d\theta \wedge dt$. For any function H in $\mathcal{F}$ we can write the associated hamiltonian vector field X_H locally on $p_{\mathcal{G}}^-1(a)$ as

$$\vartheta(t)\frac{\partial}{\partial\theta},$$

where θ is a function satisfying:

$$\int_{t_-}^{t_+} \vartheta(t)dt = H_{\mathcal{G}}(a^+) - H_{\mathcal{G}}(a^-).$$

This follows from the fact that $\omega(X_H, X_F) = 0$, so H has to be constant along the level sets of F.

Now consider a subset $D \subset U$ with smooth boundary and denote by $\langle[\phi], \partial D\rangle$ the sum of ϕ evaluated at the conjugacy classes over all boundary components of D. This value does not depend on the class $[\phi]$.

Proposition 10.7. *For an arbitrary metric with constant curvature on S, we have $\langle[\phi], \partial D\rangle = -\chi(D)$ if the boundary of D is geodesic with respect to this metric.*

Proof. First of all, since our quasimorphism ϕ is independent of ω, we can suppose that our 2-form ω is the one associated to the metric which makes the boundary ∂D geodesic. We can choose the 1-form α in such a way that it is 0 in the direction of the geodesic flow. Hence over U, for a trivialisation of the fiber bundle for which the Reeb field $X = \frac{\partial}{\partial s}$, we have $\alpha = ds + \pi^*(\lambda)$. Let γ be the orbit of a geodesic flow such that the projection $\pi(\gamma)$ is a boundary component of D. Since γ is closed we get $\phi([\pi(\gamma)]) = -\int_\gamma ds = \int_{\pi(\gamma)} \lambda$. Now by summing over all components of ∂D we get:

$$\langle[\phi], \partial D\rangle = \int_D \omega = -\chi(D)$$

□

Although for a function H in $\mathcal{F}$ an isotopy $(\varphi_H^t)_{t\in[0,1]}$ corresponding to H might not always have support in U, we can repeat the arguments in the proof of proposition 10.4. Therefore choose an element x in U, such that $p_{\mathcal{G}}(x)$ is not a vertex in $\mathcal{G}$ and denote by $[x]$ the free homotopy class corresponding to the loop $p_{\mathcal{G}}^{-1}(p_{\mathcal{G}}(x))$, oriented in the same direction as X_F. If $\widetilde{H}$ is again the function whose integral over S is 0 and which differs from H only by a constant, we have almost everywhere in U:

$$\widehat{Ang}_{\varphi_H^1}(x) = \mathfrak{M}(y \to \lambda(X_H)(y) + \widetilde{H}(y))(x) - \vartheta(x)\phi([x])$$

(where $\vartheta(x)$ is the abbreviation for $\vartheta(F(x))$). When we integrate the function $\mathfrak{M}(y \to \lambda(X_H)(y) + \widetilde{H}(y))$ over S we get $\int_S H\omega - (2g-2)H(x_l)$. For an arc a of $\mathcal{G}$ we denote with $[a]$ the value of the common class $[x]$ for $x \in p_{\mathcal{G}} - 1(a)$. Then we have

$$\int_S \vartheta(x)\phi([x])\omega = \sum_a \phi([a])(H_{\mathcal{G}}(a^+) - H_{\mathcal{G}}(a^-)).$$

where the sum goes over all arcs in $\mathcal{G}$. However, if we instead sum over all vertices of $\mathcal{G}$ we get

$$\sum_v C(v) H_{\mathcal{G}}(v),$$

where we still have to determine the exact value of $C(v)$ for each vertex. We do that in the following.

Remark 10.8 (Observation)**.** When we have two closed paths x and y with the same base point in $\pi^-1(U)$, tangent to the horocyclic foliation of S, then $\phi([\pi(x \star y)]) = \phi([\pi(x)]) + \phi([\pi(y)])$.

Using this observation we can calculate the values $C(v)$ by a case distinction as follows:

Case 1: v is a local extremum different from the global maximum x_l: Then $C(v)$ must be 0 because the value is equal to the value of ϕ on an arbitrary small closed path encircling v. Since this path is entirely inside of U it is contractible, so null-homotopic in $\pi_1(U)$.

Case 2: $v = x_l$: Here the value $C(v)$ is equal to the value of ϕ evaluated at the homotopy class in U of a small closed path encircling x_l (oriented in the opposite direction than the boundary of the subset $F \leq \lambda_l - \varepsilon$). One can easily determine that this value must be $-(2g-2)$, for example by interpreting U as the surface S punctured at x_l.

Hence we are left to determine the value $C(v)$ for vertices v which correspond to critical points of index 1. First of all note that we have

$$C(p_{\mathcal{G}}(x_j)) = \langle [\phi], \partial\{\lambda_j - \varepsilon \leq F \leq \lambda + \varepsilon\}\rangle$$

(for small enough ε). Moreover note that the domain $\{\lambda_j - \varepsilon \leq F \leq \lambda + \varepsilon\}$ consists of finitely many cylinders, for whose boundary the homogeneous quasimorphism ϕ evaluates to 0, and a single "pair of pants" P (homeomorphic to the three-holed sphere). We hence just have to evaluate $\langle [\phi], \partial P\rangle$:

Case 3: One of the three components of ∂P is null-homotopic in U.: In this case ϕ evaluates to zero on this component. Considering the other two components note that they are freely homotopic in U and the values of ϕ under the correctly oriented paths corresponding to these components differ exactly by a sign. Therefore we have $C(v) = 0$ as soon as one of the components in ∂P is null-homotopic.

Case 4: None of the components of ∂P is null-homotopic in S: One can proof, that there is a metric of constant curvature (with the same 2-form ω) which makes the boundary of P into geodesics (or at least two of the three boundary components if two of them are freely homotopic). With a similar argumentation as in proposition 10.7 we can hence deduce that $C(v) = 1$ in this case.

Case 5: At least one of the three components of ∂P is null-homotopic in S.: Denote the component which is null-homotopic in S by α_1 and the other two by α_2 and α_3. Since α_1 is not null-homotopic in U we must have that it encircles x_l, therefore neither α_2 nor α_3 are null-homotopic in S. We can modify our pair of pants P into a different one, called P', for which the boundary components $\alpha_1', \alpha_2', \alpha_3'$ are freely homotopic to the corresponding one of P and such that α_1' and α_2' share the same base point. Moreover the path α_1' can be chosen to be contained in an arbitrarily small neighborhood of x_l. We can once more find a metric of constant curvature which makes α_2' into a geodesic. Denote by β_2 the orbit periodic to the geodesic flow such that $\pi(\beta_2) = \alpha_2'$. Now we can find a closed lift β_1 of α_1', which is tangent to the horocyclic foliation, starting from the same point as β_2. Considering our observation above we hence get $\phi([\alpha_1' * \alpha_2']) = \phi([\alpha_1']) + \phi([\alpha_2'])$. Since the last component of $\partial P'$ defines the conjugacy class $[\alpha_1' * \alpha_2']^{-1}$, we therefore have $\langle [\phi], \partial P'\rangle = 0$, so also $C(v) = 0$.

Now observe those vertices v for which $C(v) = 1$ holds, are exactly the vertices in $\mathcal{V}$ defined in the previous section. In other words, the vertices in V correspond exactly to those critical points of index 1 for which none of the boundary components of the associated pair of pants P is null-homotopic in S. Moreover, recall that $C(x_l) = -(2g-2)$. To sum up we get

$$\sum_{v\in\mathcal{V}} H_{\mathcal{G}}(v) - (2g-2)H(x_l)$$

.

Combining this with our previous result therefore leads to:

$$\mathfrak{Cal}_S(\phi_H^1) = \int_S \widehat{Ang}_{\varphi_H^1}\,\omega = \int_S H\omega - \sum_{v\in\mathcal{V}} H_{\mathcal{G}}(v)$$

This concludes the proof of the theorem. □

Talk 11: A Calabi quasimorphism on $\mathrm{Ham}(S^2)$

Adrian Dawid, Reto Kaufmann

A. Introduction and Main Result

The goal of todays talk is twofold. The main goal is to show that there exists a Calabi quasimorphism on the sphere. On the other hand, we want to give a glimpse at the symplectic machinery that is required to do this proof. In the talks of the last few weeks, there was always a shortcut or some smart, combinatorial way around these powerful, but difficult concepts of Quantum homology and (filtered) Floer homology. For the sphere however, up to today, there is no known proof that does not rely upon these structures. Of course, these are topics that could fill hours and since we only have 60 minutes, we will only be able to sketch some of the most important ideas and do not even pretend to give complete proofs. The exposition follows quite closely the one in [EP03].

As said above, the main goal is to argue for the following theorem:

Theorem 11.1. *There exists a homogeneous Calabi quasimorphism* $\mu : \mathrm{Ham}(S^2) \to \mathbb{R}$.

Notation. In order to improve lisibility, we will adapt a shorthand notation and write $G = \mathrm{Ham}(S^2)$. We will further denote the identity element sometimes as e, but sometimes also as id depending on the context. Finally, to give our treatment a more general appearance, we will often talk about M but mean almost always S^2.

As seen several times in previous talks, it proves to be fruitful (and more importantly it is easier) to first find a quasimorphism on the universal cover. We will thus argue in two steps:

(1) Find a homogeneous Calabi quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ on the universal cover $\tilde{G}$ and
(2) show that it descends to a homogeneous Calabi quasimorphism $\mu : G \to \mathbb{R}$.

The first step is actually a special case of a more general theorem:

Theorem 11.2. *Let (M, ω) be a closed connected spherically monotone symplectic manifold. Suppose that the quantum homology algebra $QH_{ev}(M)$ is semi-simple. Then there exists a homogeneous Calabi quasimorphism* $\tilde{\mu} : \widetilde{\mathrm{Ham}(M)} \to \mathbb{R}$.

Remark 11.3.

- A closed connected symplectic manifold is spherically monotone if some algebraic condition involving the first Chern class of the symplectic bundle $TM \to M$ is met. We will not look into this any further than saying that the sphere S^2 is spherically monotone, but for the interested reader we will nevertheless state the formal definition:

  Definition 11.4. A closed connected symplectic manifold (M, ω) is called **spherically monotone** if there exists a real constant $\kappa > 0$ such that

  $$(c_1(M), A) = \kappa \cdot ([\omega, A]) \quad \text{for all} \quad A \in \pi_2(M)$$

  where $c_1(M)$ is the first Chern class of the symplectic bundle $TM \to M$ equipped with an ω-compatible almost complex structure J on M.

- Adrian will hint at what the quantum homology algebra is later in this talk, but for now we just need the result for $M = S^2$. The field k considered for quantum homology is $\mathbb{C}[[s]$ whose elements are formal Laurent series of the form $\sum_{j=-\infty}^{M} z_j s^j$ where $M \in \mathbb{Z}$, $z_j \in \mathbb{C}$ and s is a formal variable. The Quantum homology algebra for the sphere is then given as

  $$\begin{aligned} QH_{ev}(S^2) &= {}^{k[P]}\!\big/_{\{P^2 = s^{-1}\}} \\ &= \left\{ \sum_{j=-\infty}^{M} (z_j + Pw_j)s^j \mid M \in \mathbb{Z},\ z_j, w_j \in \mathbb{C},\ P^2 = s^{-1} \right\} \end{aligned}$$

  In particular we would like to point out that this is a field.

- A commutative algebra Q over a given field k is semi-simple if it splits into a directs sum of fields $Q = Q_1 \oplus ... \oplus Q_d$ satisfying some conditions. Again, we will not look further into this because in our case $QH_{ev}(S^2)$ is a field and will automatically satisfy these conditions. Yet again, for the interested reader we state the formal definition:

  Definition 11.5. A commutative algebra Q over a field k is called **semi-simple** if it splits into a direct sum of fields as follows:
  $$Q = Q_1 \oplus ... \oplus Q_d$$
  where
  - each $Q_i \subset Q$ is a finite-dimensional linear subspace over k,
  - each Q_i is a field with respect to the induced ring structure and
  - the multiplication in Q respects the splitting:
  $$(a_1, ..., a_d) \cdot (b_1, ..., b_d) = (a_1 b_1, ..., a_d b_d)$$

Alltogether this means that we can apply Theorem 11.2 to $M = S^2$ and that there is a homogeneous Calabi quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$. The explicit construction relies upon the heavy symplectic machinery that we are yet to discover and will be explained later.

For the second step, we now show that $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ descends to a quasimorphism $\mu : G \to \mathbb{R}$. Before doing this, we profit from the occasion and refresh our memory about the universal cover of a topological group.

Recall. Let G be a topological group and denote by $e \in G$ the identity element. For any $g \in G$ define
$$\begin{aligned} P_g &= \{\text{Paths from } e \text{ to } g\} \\ &= \{\gamma : [0,1] \to G \mid \gamma(0) = e,\ \gamma(1) = g\}. \end{aligned}$$
We will consider two paths $\gamma, \delta \in P_g$ as equivalent if they are homotopic with fixed endpoints:
$$\gamma \sim \delta \quad \text{if} \quad \gamma \simeq_{rel\,\{0,1\}} \delta$$
The universal cover of G is then the set
$$\tilde{G} := \bigsqcup_{g \in G} C_g/\sim$$
together with the obvious projection map
$$\begin{aligned} p : \tilde{G} &\to G \\ [\gamma] &\mapsto \gamma(1). \end{aligned}$$

There is a natural multiplication on $\tilde{G}$. Suppose that $\tilde{g}, \tilde{h} \in \tilde{G}$ are represented by two paths $\{g_t\}_{t\in[0,1]}$ and $\{h_t\}_{t\in[0,1]}$ from the identity e to $g = p(\tilde{g})$ and $h = p(\tilde{h})$ respectively. Then $\{g_t \cdot h_t\}_{t\in[0,1]}$ is a path from $g_0 \cdot h_0 = e \cdot e = e$ to $g_1 \cdot f_1 = f \cdot g$. The product of $\tilde{g}$ and $\tilde{h}$ is then defined as the homotopy class (relative endpoints) of this path:
$$\tilde{g} \cdot \tilde{h} = [\{g_t \cdot h_t\}_{t\in[0,1]}].$$
In particular, we can see that since
$$\ker(p) = C_e/\sim = \pi_1(G).$$
and the covering map is obviously surjective we obtain
$$\tilde{G}/\pi_1(G) \cong G.$$

We will conclude this (not so short) recall by investigating when two elements in $\tilde{G}$ commute. We start by stating a formula which is rather intuitive but that we are not going to prove formally here:
$$\{g_t \cdot h_t\}_{t\in[0,1]} \simeq_{rel\,\{0,1\}} \{h_t\}_{t\in[0,s]} * \{g_t \cdot h_s\}_{t\in[0,1]} * \{g_1 \cdot h_t\}_{t\in[s,1]} \quad \text{for all} \quad s \in [0,1].$$

where $*$ denotes the usual concatenation of paths. Consider now the two special cases $s=0$ and $s=1$ with the roles of g_t and h_t relatively inversed:

$$s=0: \qquad \{g_t \cdot h_t\}_{t\in[0,1]} \simeq_{rel\,\{0,1\}} \{g_t\}_{t\in[0,1]} * \{g_1 \cdot h_t\}_{t\in[0,1]}$$
$$s=1: \qquad \{h_t \cdot g_t\}_{t\in[0,1]} \simeq_{rel\,\{0,1\}} \{g_t\}_{t\in[0,1]} * \{h_t \cdot g_1\}_{t\in[0,1]}$$

If we now suppose that g_1 commutes with h_t for all $t \in [0,1]$, then we get using these special cases that

$$\begin{aligned}\tilde{g}\cdot\tilde{h} &= [\{g_t \cdot h_t\}_{t\in[0,1]}]\\ &= [\{g_t\}_{t\in[0,1]} * \{g_1 \cdot h_t\}_{t\in[0,1]}]\\ &= [\{g_t\}_{t\in[0,1]} * \{h_t \cdot g_1\}_{t\in[0,1]}]\\ &= [\{h_t \cdot g_t\}_{t\in[0,1]}]\\ &= \tilde{h}\cdot\tilde{g}.\end{aligned}$$

In particular, this is true if $g_1 = e$ meaning nothing but $\tilde{g} \in \pi_1(G)$. Hence the fundamental group of G lies in the centre of its universal covering group $\tilde{G}$

$$\pi_1(G) < Z(\tilde{G}),$$

a result that we will need in the proof of the next proposition.

Lemma 11.6. *Any homogeneous quasimorphism on a finite group is identically zero.*

Proof. Let $\overline{\mu} : G \to \mathbb{R}$ be a homogeneous quasimorphism on a finite group G. As G is finite, for every $g \in G$ there exists a positive integer n such that $g^n = e$. Hence

$$0 = \overline{\mu}(e) = \overline{\mu}(g^n) = n\overline{\mu}(g)$$

which implies that $\overline{\mu}(g) = 0$. □

Proposition 11.7. *The homogeneous quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ descends to a homogeneous quasimorphism $\mu : G \to \mathbb{R}$.*

Proof. Since $\pi_1(G)$ is in the centre of $\tilde{G}$, proposition 1.8 implies that for $\phi \in \pi_1(G)$ and $\tilde{f} \in \tilde{G}$ it holds that

$$\tilde{\mu}(\phi\tilde{f}) = \tilde{\mu}(\phi) + \tilde{\mu}(\tilde{f}).$$

In our case, $G = \mathrm{Ham}(S^2)$, we have [Pol12]

$$\pi_1(G) \cong \mathbb{Z}/2\mathbb{Z}.$$

so in particular $\pi_1(G)$ is finite and by lemma 11.6 we have $\tilde{\mu}(\phi) = 0$. Thus we have $\tilde{\mu}(\phi\tilde{f}) = \tilde{\mu}(\tilde{f})$ showing that there is a well-defined map

$$\begin{aligned}\mu : G \cong \tilde{G}/\pi_1(G) &\to \mathbb{R}\\ \pi_1(G)\tilde{f} &\mapsto \tilde{\mu}(\tilde{f}).\end{aligned}$$

Finally, we note that μ is homogeneous since $\tilde{\mu}$ is homogeneous:

$$\mu((\pi_1(G)\tilde{f})^n) = \mu(\pi_1(G)\tilde{f}^n) = \tilde{\mu}(\tilde{f}^n) = n\tilde{\mu}(\tilde{f}) = n\mu(\pi_1(G)\tilde{f}).$$

□

Remark 11.8. Note that this result goes through for any topological group G as long as $\pi_1(G)$ is finite.

What remains is to discuss is whether this quasimorphism is a Calabi quasimorphism. To do so, we will need its explicit expression and therefore have to introduce the symplectic machinery first.

B. Symplectic Machinery

We will now introduce some of the necessary machinery from symplectic topology. The invariant in question is called *Floer*[2] *homology.* Giving a rigorous definition would surely take much longer than the allotted time for this talk, therefore the main goal is to give a summary of the concept and gain some intuition about the key ideas behind Floer homology. In the following we will only look at the case $M = S^2$. Also as always ω is a symplectic form on $M = S^2$. Everything can be generalized to more general symplectic manifolds, but some technical changes have to be made.

C. The Setup

A leading role in the story of Floer homology will be played by the contractible loops in M.[3]

$$\Lambda = \{x \in C^\infty(S^1, M) \mid x(S^1) \sim pt\}$$

The condition of being contractible is of course equal to the loop bounding a disk in M. In the following we will want to consider loops together with (an equivalence class) of such disks. This leads us to define

$$\tilde{\Lambda} = {\{(x, u) \mid x \in \Lambda, u : D^2 \hookrightarrow M, u\mid_{\partial D^2} = x\}}\big/{\sim}.$$

Where we quotient by the relation $\sim$ defined by $(x, u_1) \sim (x, u_2) :\iff u_1\#(-u_2) = 0 \in \pi_2(M)$. The obvious projection $\tilde{\Lambda} \to \Lambda$ makes $\tilde{\Lambda}$ a covering of Λ. In our case we can see some elements of this space quite visually: We split the sphere along any loop $x \in C^\infty(S^1, M)$, then we get two elements $[x, u_1] \neq [x, u_2]$ in $\tilde{\Lambda}$. We can imagine them as representing the two components of the cut-up sphere. The next step in our recipe is a Hamiltonian function. However we will only consider Hamiltonians that satisfy a certain compatibility condition. We collect them in the set $\mathfrak{F}$.

$$\mathfrak{F} = \{F \in C^\infty(S^1 \times M, \mathbb{R}) \mid \int_M F(\cdot, t)\omega^n = 0\}$$

To finish our setup we need one more thing: The so-called action functional. For a Hamiltonian $F \in \mathfrak{F}$ it is defined by

$$\mathcal{A}_F(x, u) = \int_{S^1} F(x(t), t)dt - \int_u \omega$$

where $x \in \Lambda, u : D^2 \hookrightarrow M$ with $u\mid_{\partial D^2} = x$.

Lemma 11.9. *The functional $\mathcal{A}_F$ descends to a functional $\mathcal{A}_F : \tilde{\Lambda} \to \mathbb{R}$.*

Proof. Assume (x, u_1) and (x, u_2) represent the same class in $\tilde{\Lambda}$. Then we have $u_1\#(-u_2) = 0 \in \pi_2(M)$ which implies

$$\int_{u_1\#(-u_2)} \omega = \int_{u_1} \omega - \int_{u_2} \omega = 0 \implies \int_{u_1} \omega = \int_{u_2} \omega.$$

This already gives us $\mathcal{A}_F(x, u_1) = \mathcal{A}_F(x, u_2)$ which concludes the proof. □

To study the properties of $\mathcal{A}_F$ we introduce a useful operation on $\tilde{\Lambda}$. First we name the generator of $\pi_2(M)$, i.e. $\pi_2(M) = \mathbb{Z}_{\langle S\rangle}$. Then we define

$$s : \tilde{\Lambda} \to \tilde{\Lambda}$$
$$[(s, u)] \mapsto [x, u\#S].$$

This gives us the following property of $\mathcal{A}_F$ for any $F \in \mathfrak{F}$.

$$\begin{aligned}\mathcal{A}_F(s[x, u]) &= \int_{S^1} F(x(t), t)dt - \int_{u\#S} \omega = \int_{S^1} F(x(t), t)dt - \int_u \omega - \int_S \omega \\ &= \underbrace{\int_{S^1} F(x(t), t)dt -}_{=\mathcal{A}_F([x,u])} \int_S \omega = \mathcal{A}_F([x, u]) - \text{area}_\omega(S^2).\end{aligned}$$

[2]Named after Andreas Floer (*1956 - †1991)

[3]Since $\pi_1(M) = \pi_1(S^2) = 0$ that's all loops in our case, but for more general manifolds the restriction is important.

Now we have all the ingredients for Floer homology. So what's next? Floer described his idea in 1985 to Clifford Taubes by saying "I can do infinite-dimensional Morse theory"[4]. The basic idea of Morse homology is to compute the homology of a manifold by taking a suitable function on it and building a chain complex using its critical points. We will go forth in the same spirit. For that we need to find out what the critical points of $\mathcal{A}_F$ are.

Lemma 11.10. *For $F \in \mathfrak{F}$ we have that*

$$Crit\mathcal{A}_F = \{[x,u] \in \tilde{\Lambda} \mid \dot{x}(t) = X_H(x(t)) \forall t \in S^1\},$$

where X_H is the Hamiltonian vector field defined by F.

Proof. We can show this using a classical variational argument. Let $x_{s s\in(-\varepsilon,\varepsilon)}$ be a smooth variation of x with $\partial_s x_s|_{s=0} = \eta \in \Gamma(x^*TM)$ and u_s a corresponding variation of u. Then we have

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{A}_F([x_s,u_s])\big|_{s=0} &= \int_{S^1}\frac{\partial}{\partial s}F(x(t),t)dt\big|_{s=0} - \int_{S^1}\omega(\dot{x}(t),\eta(t))dt \\ &= \int_{S^1} dF_{x(t)}(\partial_s x(t)|_{s=0})dt - \int_{S^1}\omega(\dot{x}(t),\eta(t))dt \\ &= \int_{S^1}\omega(X_F,\eta)dt - \int_{S^1}\omega(\dot{x}(t),\eta(t))dt = \int_{S^1}\omega(X_F - \dot{x}(t),\eta)dt,\end{aligned}$$

which implies our lemma. A derivation for the derivative of the integral of ω over u_s can be found in [Flo88]. □

So we know that the 1-periodic orbits of the Hamiltonian vector field X_F are the critical points of $\mathcal{A}_F$. We will henceforth use the notation $\tilde{\mathcal{P}}_F$ for them.

Definition 11.11. We call the critical values of $\mathcal{A}_F$ the *spectrum* of $\mathcal{A}_F$ and denote them by $\operatorname{spec}\mathcal{A}_F$.

Corollary 11.12. *The set* $\operatorname{spec}\mathcal{A}_F \subset \mathbb{R}$ *is* $\operatorname{area}_\omega(S^2)\mathbb{Z}$ *invariant.*

We will use without proof that $\operatorname{spec}\mathcal{A}_F$ is closed.

D. The (Filtered) Floer Chain Complex

Now we are ready to define a chain complex.

Definition 11.13. Let $\alpha \in \mathbb{R}\cup\{\infty\}$ with $\alpha \notin \operatorname{spec}\mathcal{A}_F$, then we define

$$C_\alpha(F) := \left\{\sum_{y\in\tilde{\mathcal{P}}_F} z_y y \mid z_y \in \mathbb{C}, \mathcal{A}_F(y) < \alpha, \#\{y \mid z_y \neq 0, \mathcal{A}_F(y) > \delta\} < \infty \forall\delta\in\mathbb{R}\right\}.$$

These groups are graded by an index called the *Conley-Zehnder index* denoted $\mu_{CZ} : \tilde{\mathcal{P}}_F \to \mathbb{Z}$. While a definition of this index would be too much for the scope of this talk, the idea is as follows: In Morse homology the number of negative eigenvalues of the Hessian at a critical point is used as a grading. In the infinite-dimensional case this index would usually be infinite and thus useless. The Conley-Zehnder index is a kind of "relative Morse index". It is defined by setting it to zero for a special operator that could play the role of the Hessian in a setup like ours. Then for any other operator of this form it is possible to count how many eigenvalues change sign on a "generic" path to the reference operator and this count is finite. This procedure allows to define the Conley-Zehnder index of critical points of $\mathcal{A}_F$. More information can be found in [APS76][RS95]. Now that we have a grading the next thing we need is a differential:

Definition 11.14. (informal) The differential $d : C_\infty(F) \to C_\infty(F)$ is defined by a (signed) count of negative gradient flow lines from $x \in \tilde{\mathcal{P}}_F$ with $\mu_{CZ}(x) = k$ to $y \in \tilde{\mathcal{P}}_F$ with $\mu_{CZ}(y) = k-1$.

[4]This story was recalled by Helmut Hofer in a talk called "The Floer Jungle: 35 years of Floer Theory" held in the joint IAS/Princeton/Montreal/Paris/Tel-Aviv Symplectic Geometry seminar in July 2021. (Video recording)

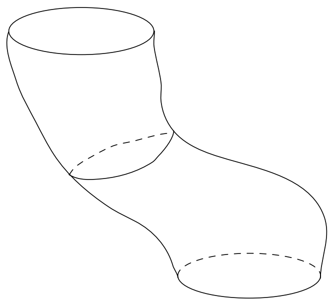


FIGURE 19. A negative gradient flow line connecting two Hamiltonian orbits.

Since we count **negative** gradient flow lines, we get the following corollary:

Corollary 11.15. *The differential $d : C_\infty(F) \to C_\infty(F)$ preserves the filtration.*

Please note that our little variational argument from above does not give us any way to make sense of the statement *negative gradient flow line* of $\mathcal{A}_F$. Making sense of that is part of Floer's seminal contribution, but would again be too much for the scope of this talk. We just note that these gradient flow lines are maps $u : S^1 \times \mathbb{R} \to M$ which converge to loops in $\tilde{\mathcal{P}}_F$ as the second parameter goes to $\pm\infty$. A visualization can be found in figure 19. The following theorem is at the heart of Floer homology:

Theorem 11.16.

$d^2 = 0.$

A proof of this theorem is quite difficult and uses many subtle analytic ideas, yet the basic concept can be seen visually. The basic idea of the proof is to realize that d^2 counts

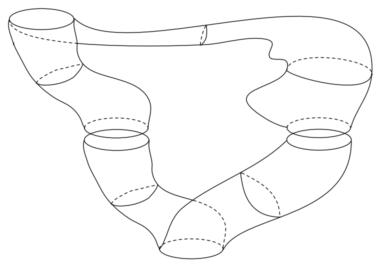


FIGURE 20. Broken flowlines being counted by d^2.

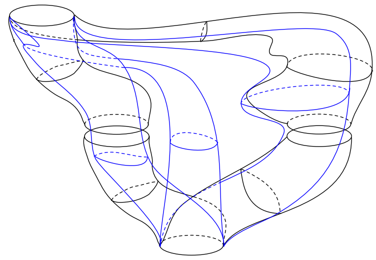


FIGURE 21. A family of flowlines connecting two broken flowlines.

the number of "broken flowlines" between orbits of the relevant Conley-Zehnder indices, cf. figure 20. The proof strategy is then, to show that families of real flowlines converge to broken flowlines. This compactness result then allows the conclusion that d^2 counts (with correct signs) the number of boundary points of a 1-manifold. Since that is 0 the result will follow. This is of course just hand-waving, but gives some intuition on why $d^2 = 0$ holds true. The full details of this argument can be found in chapter 6 of [AD14]. Figure 21 illustrates the argument. Now we finally have Floer homology:

Definition 11.17. For any $\alpha \in \mathbb{R} \setminus \operatorname{spec} \mathcal{A}_F \cup \{\infty\}$ we call the homology groups

$$V_\alpha(F) = H_{ev}(C_\alpha, d)$$

the *filtered Floer homology* of $F \in \mathfrak{F}$.

We should note that a lot of choices on which (C_α, d) a priori depends have been hidden here and that the definition above only is possible for a generic $F \in \mathfrak{F}$. However the homology is independent of these hidden choices and can be defined for all $F \in \mathfrak{F}$ by a canonical continuation procedure. Also note that the operation s defined above makes V_∞ into a k-vector space, where k is the field from Reto's talk. If we have two Hamiltonians $F, F \in \mathfrak{F}$ that generate the same element $f \in \widetilde{\mathrm{Ham}}(M)$, then there is an isomorphism between $V_\alpha(F) \cong V_\alpha(F')$ that preserves the grading. Therefore we just write $V_\alpha(f)$. We now state a theorem that justifies all our effort:

Theorem 11.18. *There exists a k-vector space isomorphism*

$$V_\infty(f) \to QH_{ev}(M) \cong H_{ev}(M; \mathbb{C}) \otimes_{\mathbb{C}} k$$

that preserves the grading.

E. The Spectral Invariants

As a last part of the symplectic machinery we will now define the spectral invariants. For this we take any $f \in \widetilde{\mathrm{Ham}}(M)$ and then use the isomorphism $QH_{ev}(M) \cong V_\infty(f)$. Now we note that for any suitable $\alpha < \beta \le \infty$ we have the natural inclusion

$$C_\alpha(f) \xrightarrow{\quad i_{\beta\alpha} \quad} C_\beta(f) \xrightarrow{\quad i_{\infty\beta} \quad} C_\infty(f).$$

This map induces a map on homology, thus leading to the following situation:

$$V_\alpha(f) \xrightarrow{\quad i^*_{\beta\alpha} \quad} V_\beta(f) \xrightarrow{\quad i^*_{\infty\beta} \quad} V_\infty(f) \xrightarrow{\quad \cong \quad} QH_{ev}(M).$$

The idea is now to view an element $a \in QH_{ev}(M)$ as the corresponding Floer homology class and track when it shows up in the filtration. Formalizing this intuition we obtain the following definition:

Definition 11.19. For $f \in \widetilde{\mathrm{Ham}}(M)$ and $a \in QH_{ev}(M)$. With $\bar{a} \in V_\infty(f)$ being the corresponding Floer homology class to a we define

$$c(a, f) \coloneqq \inf\{\alpha \mid \bar{a} \in \mathrm{Im}(V_\alpha(f) \to^{i^*_{\infty\alpha}} V_\infty(f))\}.$$

The numbers $c(a, f)$ are called the *spectral invariants* of f.

Lemma 11.20. *The spectral invariants have the following properties*

(1) $-\infty < c(a, f) < \infty$
(2) $c(a, f) \in \operatorname{spec} f$
(3) $c(a * b, fg) \le c(a, f) + c(b, g)$

for any $a, b \in QH_{ev}(M)$ and $f, g \in \widetilde{\mathrm{Ham}}(M)$.

The proofs of these properties can be found in [EP03].

F. Quasimorphism on $\tilde{G}$

We saw earlier that in our case the quantum homology $QH_{ev}(S^2)$ is a field. Hence there is a preferred element, namely the unity e which we can feed to the spectral invariant. This then defines a function $\tilde{G} \to \mathbb{R}$. More precisely, we have

Theorem 11.21. *The function*

$$r : \tilde{G} \to \mathbb{R}$$
$$\tilde{f} \mapsto c(e, \tilde{f})$$

is a quasimorphism.

Sketch of the proof: (For simplicity we will drop the tilde for elements of $\tilde{G}$ in this proof.) We saw in Lemma 11.20 that spectral invariants satisfy a triangle inequality

$$c(a * b, fg) \le c(a, f) + c(b, g) \quad \text{for all} \quad a, b \in QH_{ev}(S^2).$$

Using now that $e = e * e$ we get that for $f, g \in \tilde{G}$

$$c(e, fg) = c(e * e, fg) \le c(e, f) + c(e, g)$$

Similarly, we write

$$c(e, f) = c(e * e, fgg^{-1}) \le c(e, fg) + c(e, g^{-1})$$

which yields by rearranging that

$$c(e, fg) \ge c(e, f) - c(e, g^{-1}).$$

Using the tools of the symplectic machinery advertised for by Adrian, we can work on the last term and write $c(e, g^{-1})$ as some infimum

$$c(e, g^{-1}) = - \inf_{b : \Pi(b, e) \neq 0} c(b, g)$$

for which we can then find (with some more work) a lower bound. Eventually, we get

$$c(e, fg) \geq c(e, f) + c(e, g) - C$$

where C is some positive real constant. In conclusion, we have argumented that

$$\begin{cases} r(fg) - r(f) - r(g) = c(e, fg) - c(e, f) - c(e, g) \leq 0 \\ r(fg) - r(f) - r(g) = c(e, fg) - c(e, f) - c(e, g) \geq -C \end{cases}$$

which finally shows that

$$|r(fg) - (r(f) + r(g))| \leq C$$

and thereby that $r : \tilde{G} \to \mathbb{R}$ is a quasimorphism. □

We have thus found a quasimorphism on $\tilde{G}$ and we can obtain from it with the usual homogenisation a homogeneous quasimorphism $\tilde{\mu} : \tilde{G} \to \mathbb{R}$, that is,

$$\tilde{\mu}(\tilde{f}) = - \lim_{m \to \infty} \frac{r(\tilde{f}^m)}{m}.$$

G. Calabi quasimorphism on $\mathrm{Ham}(S^2)$

Recall. For every non-empty open subset $U \subsetneq M$ there is a natural subgroup G_U of G consisting of all elements $f \in G$ which are generated by a time-dependent Hamiltonian $F_t : M \to \mathbb{R}$ such that $supp(F_t) \subset U$.

For each such $U \subsetneq M$, define the map

$$Cal_U : G_U \to \mathbb{R}$$
$$f \mapsto \int_0^1 \mathrm{d}t \int_M F_t \omega^n.$$

If the symplectic form ω is exact on U, this map does not depend on the choice of the Hamiltonian F_t generating f. Cal_U is a homomorphism called the **Calabi homomorphism.**

Definition 11.22. A non-empty open subset $U \subset M$ is called **displaceable** (by a Hamiltonian diffeomorphism) if there exists $h \in G = \mathrm{Ham}(M)$ such that

$$hU \cap \overline{U} = \emptyset.$$

We denote by $\mathcal{D}$ the set of displaceable subsets.

Notation. We are interested in the subset of $\mathcal{D}$ for which Cal_U is a well-defined homomorphism, that is, the subsets $U \subset M$ for which ω is exact. We will denote this by

$$\mathcal{D}_{ex} = \{U \in \mathcal{D} \mid \omega \text{ is exact on U}\}.$$

Definition 11.23. A quasimorphism on G is called a **Calabi quasimorphism** if it coincides with the Calabi homomorphism Cal_U on any $U \in \mathcal{D}_{ex}$.

Remark 11.24. Actually, we need this language adapted to the universal cover $\tilde{G}$.

- For a non-empty open subset $U \subsetneq M$ we get a subgroup $\tilde{G}_U < \tilde{G}$: $\tilde{f} \in \tilde{G}$ is in $\tilde{G}_U$ if and only if it can be represented by a Hamiltonian flow $\{f_t\}_{t \in [0,1]}$ (with $f_0 = \mathrm{Id}$) which is generated by a Hamiltonian F_t with support in U.
- Almost the same formula as above determines a well-defined homomorphism
$$\widetilde{Cal}_U : \tilde{G}_U \to \mathbb{R}$$
$$\tilde{f} \mapsto \int_0^1 \mathrm{d}t \int_M F_t \omega^n.$$
if ω is exact on U.
- A quasimorphism on $\tilde{G}$ is called a **Calabi quasimorphism** if it coincides with the Calabi homomorphism $\widetilde{Cal}_U$ on any $U \in \mathcal{D}_{ex}$.

Proposition 11.25. *Let $U \in \mathcal{D}_{ex}$, $\tilde{f} \in \tilde{G}_U$ and $\tilde{\mu} : \tilde{G} \to \mathbb{R}$ be as above. Then*

$$\tilde{\mu}(\tilde{f}) = \widetilde{Cal}_U(\tilde{f}).$$

Proof. $\tilde{f} \in \tilde{G}_U$ is by definition generated by a Hamiltonian function with support in U. Explicitely, let

$$F : M \times \mathbb{R} \to \mathbb{R}$$
$$(p,t) \mapsto F_t(p)$$

be the Hamiltonian function which is 1-periodic in time and whose support is contained in U, that is, $F_t|_{M\setminus U} = 0$. As usual, this determines a family $\{X_t\}$ of Hamiltonian vector fields and this in turn corresponds to a hamiltonian isotopy Φ_t^F. Recall that this means that Φ_t^F is a path in $\mathrm{Ham}(M)$ such that $\Phi_0^F = \mathrm{id}$ and $\Phi_1^F = f = p(\tilde{f})$, or equivalenty $\tilde{f} = [\{\Phi_t^F\}_{t\in[0,1]}]$. Note then that since F is 1-periodic in time, the same holds also for Φ_t^F and therefore we may write

$$\tilde{f}^m = [\{\Phi_t^F\}_{t\in[0,m]}] \quad \text{for all} \quad m \in \mathbb{Z}.$$

Further, we see that Φ_t^F is the identity outside of U (since $supp(F_t) \subset U$) and hence it must map U to itself $\Phi_t^F(U) = U$.

On the other hand, by definition of $\mathcal{D}_{ex}$ there exists a hamiltonian diffeomorphism $h \in G$ such that

$$h(U) \cap \overline{U} = \emptyset.$$

Write $\tilde{h} \in \tilde{G}$ for any lift of h, that is, a path $\tilde{h}$ from the identity to $h \in G$. h maps by assumption any point of U out of U and therefore the fixed points of the product $h\Phi_t$ are exactly the fixed points of h:

$$Fix(h\Phi_t^F) = Fix(h) \subset M\setminus U.$$

Next, we define

$$F' : M \times \mathbb{R} \to \mathbb{R}$$
$$(p,t) \mapsto F'_t(p) = F_t(p) \underbrace{- \int_M F_t \omega^n}_{:=u(t)}$$

and note right away that

(1) since $supp(F_t) \subset U$ this reduces to the second term $u(t)$ on $M\setminus U$.
(2) The second term $u(t)$ is 1-periodic and does not depend upon p. As they only differ by a constant, F'_t and F_t generate the same hamiltonian isotopy Φ_t^F.
(3) Finally, F' satisfies the normalisation condition $\int_M F'_t\omega^n = 0$ and is also 1-periodic.

In particular, we can now apply the machinery seen in Adrian's talk: First we note that

$$Spec(\tilde{h}\tilde{f}^m) = Spec(\tilde{h}[\{\Phi_t^F\}_{t\in[0,m]}]) = Spec(\tilde{h}) + \underbrace{\int_0^m u(t)\,\mathrm{d}t}_{:=w(m)}$$

which is important since

$$r(\tilde{h}\tilde{f}^m) = c(e, \tilde{h}\tilde{f}^m) \in Spec(\tilde{h}\tilde{f}^m) = Spec(\tilde{h}) + w(m).$$

More precisely, this means that there exists a $s_0 \in Spec(\tilde{h})$ so that

$$r(\tilde{h}\tilde{f}^m) = s_0 + w(m).$$

Now we are ready for the final computation:

$$\begin{aligned}
\tilde{\mu}(\tilde{f}) &= -\lim_{m\to\infty} \frac{r(\tilde{f}^m)}{m} \\
&= -\lim_{m\to\infty} \frac{r(\tilde{h}\tilde{f}^m)}{m} \\
&= -\lim_{m\to\infty} \frac{w(m)}{m} \\
&= \lim_{m\to\infty} \frac{1}{m} \underbrace{\int_0^m u(t)\ \mathrm{d}t}_{=m\int_0^1 u(t)\ \mathrm{d}t} \\
&= \int_0^1 \int_M F_t\omega\ \mathrm{d}t \\
&= \widetilde{Cal}_U(\tilde{f})
\end{aligned}$$

where going from the first to the second line we use that r is a quasimorphism. □

Corollary 11.26. *Let $U \in \mathcal{D}_{ex}$, $f \in G_U$ and $\mu : G \to \mathbb{R}$ be as above. Then*

$$\mu(f) = Cal_U(f).$$

Proof. This follows directly from the above proof by noting that

$$\mu(f) = \tilde{\mu}(\tilde{f}) = ... = \int_0^1 \int_M F_t\omega\ \mathrm{d}t = Cal_U(f).$$

□

Talk 12: Construction of the real numbers by quasimorphisms

Jonathan, Julian

In this talk we follow [A1].

A. Slopes and real numbers

The aim of this talk is to construct the real numbers using a special kind of quasimorphisms called slopes. Even though there are many other ways to construct the real numbers, most of them use the rational numbers to construct real numbers via some sort of completion. The approach in this talk only uses integers to construct the real numbers.

Definition 12.1. A slope is a map $\lambda\colon \mathbb{Z} \to \mathbb{Z}$ such that there exists a constant $C \geq 0$ with the property that for all $m, n \in \mathbb{Z}$

$$|\lambda(m+n) - \lambda(m) - \lambda(n)| \leq C.$$

Remark 12.2. Note that the definition of slopes is very similar to the one of quasimorphisms on the group $\mathbb{Z}$. The only further assumption we have in the definition of slopes is that the target space is $\mathbb{Z}$ instead of $\mathbb{R}$. This is of course needed, since our aim is to define a construction of $\mathbb{R}$ using slopes and we cannot do this by using quasimorphisms which have $\mathbb{R}$ as target space.

Definition 12.3. Let λ, λ' be slopes. We say that λ, λ' are equivalent if the set

$$\{\lambda(n) - \lambda'(n) \mid n \in \mathbb{Z}\}$$

is bounded.

Proposition 12.4. *The equivalence of slopes is an equivalence relation on the set of slopes.*

Definition 12.5. A real number is an equivalence class of slopes. We denote by $\mathbb{R}$ the set of real numbers.

Remark 12.6. For any $j \in \mathbb{Z}$, we can define a slope $\bar{j}\colon \mathbb{Z} \to \mathbb{Z}, n \mapsto nj$. So we can identify an integer j with the equivalence class of its corresponding slope $\bar{j}$. This identification allows us to regard $\mathbb{Z}$ as a subset of $\mathbb{R}$.

Definition 12.7. For $[\alpha], [\beta] \in \mathbb{R}$ we define

$$[\alpha] + [\beta] := [\alpha + \beta]$$

and

$$[\alpha] \cdot [\beta] = [\alpha \circ \beta]$$

Lemma 12.8. *The above operations are well-defined.*

Proof. To show that the operations are well-defined, we need to show that the sum and the compositions of two slopes are slopes again and that the definitions of $+$ and $\cdot$ are independent of the choice of slopes in the equivalence classes. Let α and β be slopes. Denote by C_α the defect of α and by C_β the defect of β. Then we have that for all $m, n \in \mathbb{Z}$

$$\begin{aligned}
&|(\alpha+\beta)(m+n) - (\alpha+\beta)(m) - (\alpha+\beta)(n)| \\
&= |\alpha(m+n) - \alpha(m) - \alpha(n) + \beta(m+n) - \beta(m) - \beta(n)| \\
&\leq |\alpha(m+n) - \alpha(m) - \alpha(n)| + |\beta(m+n) - \beta(m) - \beta(n)| \\
&\leq C_\alpha + C_\beta
\end{aligned}$$

so $\alpha + \beta$ is a slope.

Also we have that

$$\begin{aligned}
&|(\alpha\circ\beta)(m+n)-(\alpha\circ\beta)(m)-(\alpha\circ\beta)(n)|\\
=&\ |\alpha(\beta(m+n))-\alpha(\beta(m))-\alpha(\beta(n))|\\
\le&\ |\alpha(\beta(m+n))-\alpha(\beta(m)+\beta(n))|+|\alpha(\beta(m)+\beta(n))-\alpha(\beta(m))-\alpha(\beta(n))|\\
\le&\ |\alpha(\beta(m+n))-\alpha(\beta(m)+\beta(n))|+C_\alpha\\
\le&\ |\alpha(\beta(m+n)-\beta(m)-\beta(n)+(\beta(m)+\beta(n)))-\alpha(\beta(m+n)-\beta(m)-\beta(n))-\alpha(\beta(m)+\beta(n))|\\
&\ +|\alpha(\beta(m+n)-\beta(m)-\beta(n))|+C_\alpha\\
\le&\ 2C_\alpha+|\alpha(\beta(m+n)-\beta(m)-\beta(n))|\\
\le&\ 2C_\alpha+max\{|\alpha(r)|\mid r\in\mathbb{Z},\ |r|\le C_\beta\},
\end{aligned}$$

which shows that $\alpha\circ\beta$ is indeed a slope.

Let α' be a slope which is equivalent to the slope α and β' be a slope which is equivalent to the slope β. Denote by A a bound of $\alpha-\alpha'$ and by B a bound of $\beta-\beta'$
Then it follows that for all $n\in\mathbb{Z}$

$$\begin{aligned}
|(\alpha+\beta)(n)-(\alpha'+\beta')(n)| &= |\alpha(n)+\beta(n)-\alpha'(n)-\beta'(n)|\\
&= |\alpha(n)-\alpha'(n)+\beta(n)-\beta'(n)|\\
&\le |\alpha(n)-\alpha'(n)|+|\beta(n)-\beta'(n)|\le A+B.
\end{aligned}$$

This shows that $\alpha+\beta$ and $\alpha'+\beta'$ are equivalent slopes and therefore the addition is well-defined on $\mathbb{R}$.
Also we obtain that

$$\begin{aligned}
|(\alpha\circ\beta)(n)-(\alpha'\circ\beta')(n)| &= |\alpha(\beta(n))-\alpha'(\beta'(n))|\\
&= |\alpha((\beta(n)-\beta'(n))+\beta'(n))-\alpha'(\beta'(n))|\\
&\le |\alpha((\beta(n)-\beta'(n))+\beta'(n)))-\alpha((\beta(n)-\beta'(n))-\alpha(\beta'(n))|\\
&\quad +|\alpha((\beta(n)-\beta'(n))|+|\alpha(\beta'(n))-\alpha'(\beta'(n))|\\
&\le C_\alpha+|\alpha((\beta(n)-\beta'(n))|+A\\
&\le C_\alpha+max\{|\alpha(r)|\mid |r|\le B,\ r\in\mathbb{Z}\}+A,
\end{aligned}$$

which shows that $\alpha\circ\beta$ and $\alpha'\circ\beta'$ are equivalent slopes. So the multiplication is also well-defined. □

Lemma 12.9. *Let λ be a slope and C_λ be the defect of λ then for any $n,\ k\in\mathbb{Z}$ it holds that*

$$|\lambda(kn)-k\lambda(n)|\le|k|C_\lambda.$$

Proof.

$$\begin{aligned}
|\lambda(kn)-k\lambda(n)| &= |\lambda(n+(k-1)n)-\lambda(n)-(k-1)\lambda(n)|\\
&\le |\lambda(n+(k-1)n)-\lambda(n)-\lambda((k-1)n)|+|\lambda((k-1)n)-(k-1)\lambda(n)|\\
&\le C_\lambda+|\lambda((k-1)n)-(k-1)\lambda(n)|\le\cdots\le|k|C_\lambda
\end{aligned}$$

□

Proposition 12.10. *The triple $(\mathbb{R},+,\cdot)$ is a unitary commutative ring and $1\neq 0$ in $\mathbb{R}$.*

Proof. We first show that multiplication is commutative. Let $[\alpha],[\beta]\in\mathbb{R}$. We want to show that $\alpha\circ\beta$ and $\beta\circ\alpha$ are equivalent. Let C_α be the defect of α and C_β be the defect of β. Using Lemma 12.9 for $n\in\mathbb{Z}\backslash\{0\}$ we obtain that

$$n\alpha(\beta(n))-\alpha(n\beta(n))\le|\alpha(n\beta(n))-n\alpha(\beta(n))|\le|n|C_\alpha$$

and hence we get that

$$n\alpha(\beta(n))\le\alpha(n\beta(n))+|n|C_\alpha.$$

Also note that

$$\begin{aligned}\alpha(n\beta(n)) &= \alpha(\beta(n)n)\\ &\le \alpha(\beta(n)n) - \beta(n)\alpha(n) + \beta(n)\alpha(n)\\ &\le |\alpha(\beta(n)n) - \beta(n)\alpha(n)| + \beta(n)\alpha(n)\\ &\le |\beta(n)|C_\alpha + \beta(n)\alpha(n)\end{aligned}$$

where the last inequality is obtained by applying Lemma 12.9 again. So by combining our two estimates it follows that

$$n\alpha(\beta(n)) \le \alpha(n\beta(n)) + |n|C_\alpha \le |\beta(n)|C_\alpha + \beta(n)\alpha(n) + |n|C_\alpha$$

and therefore we get

$$n\alpha(\beta(n)) - \beta(n)\alpha(n) \le |n|C_\alpha + |\beta(n)|C_\alpha.$$

Now we use the fact that

$$|\beta(n)| = |\beta(n\cdot 1)| \le |\beta(n\cdot 1) - n\beta(1)| + |n||\beta(1)| \le |n|C_\beta + |n||\beta(1)| = |n|(C_\beta + |\beta(1)|)$$

where the last inequality is obtained from Lemma 12.9. So we can conclude that

$$n\alpha(\beta(n)) - \beta(n)\alpha(n) \le |n|C_\alpha + |\beta(n)|C_\alpha \le |n|C_\alpha(1 + C_\beta + |\beta(1)|).$$

In an analogous way by just exchanging the roles of α and β we obtain that

$$n\beta(\alpha(n)) - \alpha(n)\beta(n) \le |n|C_\beta(1 + C_\alpha + |\alpha(1)|).$$

We can now use those two estimates to obtain that

$$\begin{aligned}|n||(\alpha\circ\beta)(n) - (\beta\circ\alpha)(n)| &\le |n\alpha(\beta(n)) - \beta(n)\alpha(n)| + |n\beta(\alpha(n)) - \alpha(n)\beta(n)|\\ &\le |n|C_\alpha(1 + C_\beta + |\beta(1)|) + |n|C_\beta(1 + C_\alpha + |\alpha(1)|).\end{aligned}$$

After dividing by $|n|$ we obtain that

$$|(\alpha\circ\beta)(n) - (\beta\circ\alpha)(n)| \le C_\alpha(1 + C_\beta + |\beta(1)|) + C_\beta(1 + C_\alpha + |\alpha(1)|)$$

which shows that $\alpha\circ\beta$ and $\beta\circ\alpha$ are equivalent and hence the multiplication on $\mathbb{R}$ is commutative.

The proofs of the other ring axioms are much easier. They follow directly from the definitions of $+$ and $\cdot$ and the fact that $\mathbb{Z}$ is a unitary commutative ring. We will prove distributivity. The other axioms can be shown in an analogous way. Let $[\alpha], [\beta], [\gamma] \in \mathbb{R}$. Then we have

$$\begin{aligned}([\alpha] + [\beta])\cdot[\gamma] &= [\alpha+\beta]\cdot[\gamma] = [(\alpha+\beta)\circ\gamma] = [\alpha\circ\gamma + \beta\circ\gamma]\\ &= [\alpha\circ\gamma] + [\beta\circ\gamma] = [\alpha]\cdot[\gamma] + [\beta]\cdot[\gamma].\end{aligned}$$

□

Definition 12.11. A slope λ is called positive if the set $\{\lambda(n) \mid n \in \mathbb{N},\ \lambda(n) \le 0\}$ is finite and the set $\{\lambda(n) \mid n \in \mathbb{Z}\}$ is infinite.

Definition 12.12. We say that a slope λ is less than a slope λ' if the slope $\lambda' - \lambda$ is positive.

Definition 12.13. A real number is called positive if all the slopes in the equivalence class are positive.

Definition 12.14. We say that a real number a is less than a real number b if the real number $b - a$ is positive.

B. Construction of rational numbers

Unlike other constructions of the real numbers, this construction uses $\mathbb{Z}$ instead of $\mathbb{Q}$. Hence, we also have to construct the rational numbers.
First, we shall think about the motivation behind the construction, using our intuition. Let $p, q \in \mathbb{Z}$ with $q > 0$. Then we want to represent the function $f(n) = \frac{p}{q}n$. Of course we have the problem that the image of f does not lie in $\mathbb{Z}$ if q does not divide p. Hence, we want to approximate $\frac{p}{q}n$. We do this as follows: $\phi(n) := \max\{k \in \mathbb{Z} \mid k \leq n\frac{p}{q}\}$
This is not formal as this definition relies on $\frac{p}{q}$. We shall do it formally correct:

Definition 12.15. For $p, q \in \mathbb{Z}$ with $q > 0$, define

$$\phi : \mathbb{Z} \to \mathbb{Z}$$
$$n \mapsto \begin{cases} \max\{k \in \mathbb{Z} \mid kq \leq np\} & n > 0 \\ -\phi(-n) & n < 0 \\ 0 & n = 0 \end{cases}$$

and denote $\frac{p}{q}$ as the equivalence class of ϕ.

Proposition 12.16. *This ϕ defines a slope and $\bar{q} \circ \phi$ is equivalent to $\bar{p}$ i.e. $q\frac{p}{q} = p$ as desired.*

Proof. We find for $n \geq 0$

$$q\phi(n) \leq np$$
$$q(\phi(n) + 1) > np$$

and for $n < 0$

$$q\phi(n) \geq np$$
$$q(\phi(n) - 1) < np$$

Therefore, we deduce $|q\phi(n) - np| \leq q$. We compute:

$$\begin{aligned} q|\phi(m+n) - \phi(m) - \phi(n)| &\leq |q\phi(m+n) - (m+n)p| + |q\phi(m) - mp| + |q\phi(n) - np| \\ &\leq q + q + q = 3q \end{aligned}$$

After dividing by q, we find that $|\phi(m+n) - \phi(m) - \phi(n)| \leq 3$ is bounded and ϕ is a slope.
The same estimates above show that $|\bar{q}(\phi(n)) - \bar{p}(n)|$ is bounded by q. Hence, $\bar{q} \circ \phi$ and $\bar{p}$ represent the same slope. □

C. Examples of selected real numbers

Similar to the construction of the rational numbers. We construct our slopes first on $\mathbb{Z}^{>0}$ and then extend it to $\mathbb{Z}$ with $\phi(n) = -\phi(-n)$ for $n < 0$ and $\phi(0) = 0$. Then it suffices to check that $|\phi(m+n) - \phi(m) - \phi(n)|$ is bounded for all $m, n > 0$ to show that ϕ is a slope. In the following, we shall thus do the work only for $n > 0$.

C.1. Constructing $\sqrt{2}$

Again, we have the motivation that we want to approximate $f(n) = \sqrt{2}n$ via $\rho(n) := \max\{k \in \mathbb{Z} \mid k \leq \sqrt{2}n\}$. We write this without using $\sqrt{2}$ as follows: $\rho(n) = \max\{k \in \mathbb{Z} \mid k^2 \leq 2n^2\}$. We denote the equivalence class of ρ as $\sqrt{2}$.

Proposition 12.17. *The function ρ is a slope, $\rho \circ \rho$ is equivalent to $\bar{2}$ i.e. $\sqrt{2}^2 = 2$ and the slope ρ is positive.*

Proof. For $n > 0$, we see $n^2 \leq 2n^2 \leq 4n^2$ and thus $n \leq \rho(n) \leq 2n$, therefore ρ is a positive slope (if it is a slope).
Moreover, we get $\rho(n)^2 \leq 2n^2$ (this is the usual square for integers) and $(\rho(n)+1)^2 > 2n^2$. Hence, we find

$$2(n-1)^2 = 2n^2 - 4n + 2 \leq \rho(n)^2 \leq 2n^2 \tag{40}$$

and for $n, m > 0$

$$4(n-1)^2(m-1)^2 \leq \rho(n)^2\rho(m)^2 \leq 4n^2m^2$$

and after taking the root (in $\mathbb{Z}$) we get

$$2(n-1)(m-1) \leq \rho(n)\rho(m) \leq 2nm$$

Now, to estimate $-\rho(m+n)+\rho(m)+\rho(n)$ we multiply it with $\rho(m+n)+\rho(m)+\rho(n)$, note that $2(m+n) \leq \rho(m+n)+\rho(m)+\rho(n) \leq 4(m+n)$. We compute

$$\begin{aligned} x &:= (-\rho(m+n)+\rho(m)+\rho(n))(\rho(m+n)+\rho(m)+\rho(n)) \\ &= -\rho(m+n)^2+\rho(m)^2+\rho(n)^2+2\rho(m)\rho(n) \end{aligned}$$

We can estimate:

$$\begin{aligned} -3 &\leq (-8n-8m-8)/(4(m+n)) \\ &\leq (-2(n+m)^2+2(n-1)^2+2(m-1)^2+4(n-1)(m-1)/(4(m+n)) \\ &\leq x/(\rho(m+n)+\rho(m)+\rho(n)) \\ &= -\rho(m+n)+\rho(m)+\rho(n) \\ &= x/\rho(m+n)+\rho(m)+\rho(n) \\ &\leq (-2(m+n-1)^2+2m^2+2n^2+4mn)/(2(m+n)) \\ &= (2+4(m+n))/(2(m+n)) \\ &\leq 3 \end{aligned}$$

Hence, ρ is a slope.
We note $2(\rho(k)-1)^2 \leq \rho(\rho(k))^2 \leq 2\rho(k)^2$ with (40) for $n = \rho(k)$. Therefore, we find

$$4(k-2)^2 \leq 2(\rho(k)-1)^2 \leq \rho(\rho(k))^2 \leq 2\rho(k)^2 \leq 4k^2$$

and therefore

$$2(k-2) \leq \rho(\rho(k)) \leq 2k$$

or equivalently

$$-4 \leq \rho \circ \rho(k) - \bar{2}(k) \leq 0$$

and therefore, $\rho \circ \rho$ is equivalent to $\bar{2}$. □

C.2. Constructing the root of a polynomial

Consider the polynomial $p(x) := x^5 + x - 3$. We note that $p'(x) = 5x^4 + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$. Therefore p is strictly monotonously increasing and thus as exactly one root a in $\mathbb{R}$. But with Galois theory an Abel-Ruffini one can prove that a cannot be written with radicals in $\mathbb{Q}$. However, constructing a is quite straightforward with slopes.
We define $\alpha(n) := \max\{k \in \mathbb{Z} \mid p(\frac{k}{n}) \leq 0\} = \max\{k \in \mathbb{Z} \mid k^5 + n^4k - 3n^5 \leq 0\}$.

Proposition 12.18. *The function α defines a slope and the equivalence class of α is a with $p(a)$=0 or equivalently $n \mapsto \alpha^{\circ 5}(n) + \alpha(n) - \bar{3}(n)$ is bounded.*

Notation. For $e \geq 1$, we define recursively

$$\alpha^{\circ e} = \begin{cases} \alpha \circ \alpha^{\circ(e-1)} & e > 1 \\ \alpha & e = 1 \end{cases}$$

Proof of 12.18. From the definition of α, we find $p(\frac{\alpha(n)}{n}) \leq 0 \leq p(\frac{\alpha(n)+1}{n})$.
Take $m, n > 0$. We define:

$$\begin{aligned} a_- &= \max\left\{\frac{\alpha(n)}{n}, \frac{\alpha(m)}{m}, \frac{\alpha(m+n)}{m+n}\right\} \\ a_+ &= \min\left\{\frac{\alpha(n)+1}{n}, \frac{\alpha(m)+1}{m}, \frac{\alpha(m+n)+1}{m+n}\right\} \end{aligned}$$

We find $p(a_-) \leq 0 \leq p(a_+)$. By monotonicity of p, we deduce that $a_- \leq a_+$. Now, choose any $A \in \mathbb{Q}$ such that $a_- \leq A \leq a_+$, then we find

$$\frac{\alpha(n)}{n}, \frac{\alpha(m)}{m}, \frac{\alpha(n+m)}{n+m} \leq a_- \leq A \leq a_+ \leq \frac{\alpha(n)+1}{n}, \frac{\alpha(m)+1}{m}, \frac{\alpha(n+m)+1}{n+m}$$

This shows that $|\alpha(n) - nA|, |\alpha(m) - mA|, |\alpha(n+m) - (n+m)A| \leq 1$ which gives $|\alpha(n+m) - \alpha(n) - \alpha(m)| \leq 3$ with the trangle inequality. Hence, α is a slope.
To see that $\alpha^{\circ 5} + \alpha - \bar{3}$ is bounded we use the following claim:

Claim. For all $n, e > 0$, we have

$$|n^{e-1}\alpha^{\circ e}(n) - \alpha(n)^e| \leq n^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-1}$$

with $S_\alpha := \max\{|\alpha(k+l) - \alpha(k) - \alpha(l)| \mid k, l > 0\}$.

Proof of Claim. We prove it via induction on e. For $e = 1$, we have $0 \leq 1$. For the induction step, we need the preliminary estimates

$$|\alpha(n)| \leq |\alpha(n) - \overbrace{\alpha(1) - \cdots - \alpha(1)}^{n}| + n|\alpha(1)| \leq (n-1)S_\alpha + n|\alpha(1)| \leq n(|\alpha(1)| + S_\alpha)$$

and with the triangle inequality

$$|\alpha(n) - n| \leq n(1 + |\alpha(1)| + S_\alpha).$$

Now, we can estimate

$$\begin{aligned}|n^e\alpha^{\circ e+1}(n) - \alpha(n)^{e+1}| &\leq n|n^{e-1}\alpha^{\circ e}(n) - \alpha(n)^e| + |n - \alpha(n)||\alpha(n)^{e-1}| \\ &\leq nn^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-1} + n(1 + |\alpha(1)| + S_\alpha) \\ &\leq n(1 + |\alpha(1)| + S_\alpha)(n^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-2} + 1) \\ &\leq n(1 + |\alpha(1)| + S_\alpha)(n^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-1})\end{aligned}$$

where the last inequality uses that $(1 + |\alpha(1)| + S_\alpha) \geq 2$. □

Using this claim, we can replace $\alpha^{\circ 5}$ with an equivalent and easier to handle slope, namely $\alpha^{\circ 5}$ is equivalent to $\lfloor \frac{\alpha(n)^5}{n^4} \rfloor$ and $\alpha^{\circ 5} + \alpha - \bar{3}$ is equivalent to the slope ϵ defined by

$$\epsilon(n) := \lfloor \frac{\alpha(n)^5}{n^4} \rfloor + \alpha(n) - 3n$$

We compute:

$$\begin{aligned}0 \leq \epsilon(n) &= np\left(\frac{\alpha(n)}{n}\right) \\ &= np\left(\frac{\alpha(n)-1}{n} + \frac{1}{n}\right) \\ &= n\left(p\left(\frac{\alpha(n)-1}{n}\right) + \frac{p'(\xi)}{n}\right) \\ &\leq n\left(p\left(\frac{\alpha(n)-1}{n}\right) + \frac{p'(2)}{n}\right) \leq p'(2) = 81\end{aligned}$$

where we used the mean value theorem in the third line for some $\xi \in \left(\frac{\alpha(n)-1}{n} + \alpha(n)\right)$ and we used monotonicity of p' in the last line. Hence, ϵ is bounded and α indeed represents a root of p. □

C.3. Constructing π

We want to define π using the formula that a circle with radius r has area πr^2. For this we count the lattice points which are contained in a circle of radius $\sqrt{n}$: $\beta(n) := |\{(p, q) \in \mathbb{Z} \times \mathbb{Z} \mid p^2 + q^2 \leq n\}|$.
Unfortunately, β is not a slope as one can for example observe that $\beta(5^u) - \beta(5^u - 1) - \beta(1) = 4^u - 1$. For this we define:

$$\bar{\beta}(n) := \left\lfloor \frac{\beta(n^2)}{n} \right\rfloor$$

We note that if we draw squares with centres at the lattice points, these square are contained in the circle with radius $\sqrt{n}+\frac{\sqrt{2}}{2}$, and contain the circle with radius $\sqrt{n}-\frac{\sqrt{2}}{2}$ as in figure 23. Hence $\pi(\sqrt{n}-\frac{\sqrt{2}}{2})^2 \leq \beta(n) \leq \pi(\sqrt{n}+\frac{\sqrt{2}}{2})^2$ and thus $|\beta(n)-n\pi| \leq 2\sqrt{2}\sqrt{n}\pi$.

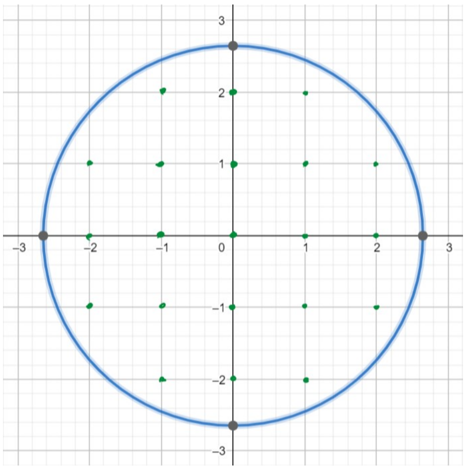


FIGURE 22. The 21 lattice points in the circle with radius $\sqrt{7}$

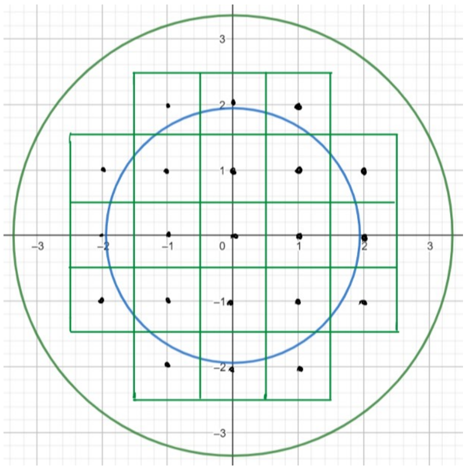


FIGURE 23. The squares are contained in the circle of radius $\sqrt{7}+\frac{\sqrt{2}}{2}$ but contains the circle of radius $\sqrt{7}-\frac{\sqrt{2}}{2}$.

This gives that $|\bar{\beta}(n)-n\pi| \leq 2\sqrt{2}\pi+1$ which not only shows that $\bar{\beta}$ represents π but also that $\bar{\beta}$ is a slope.

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