The arc-Floer conjecture for isolated homogeneous singularities

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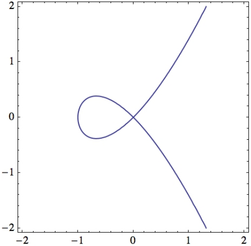

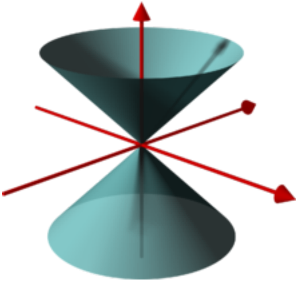

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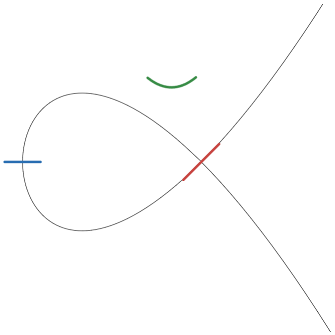

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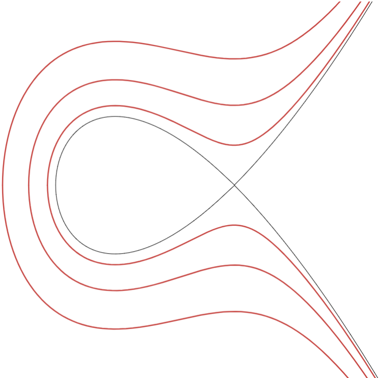

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- the Milnor fiber – a nearby smooth locus of the singularity (of topological natrue)
- The arc-Floer conjecture:

$$\text{the cohomology of contact loci} \quad \cong \quad \text{Floer homology on Milnor fiber}$$

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storing degree 1 infinitesimal information (a tangent direction).

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Definition

The *m-th restricted contact locus* is

$$\mathcal{X}_m := \left\{ \gamma : \operatorname{Spec}\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}^n \;\middle|\; \begin{array}{l} \gamma(0) = 0, \\ f(\gamma(t)) = t^m \bmod t^{m+1} \end{array} \right\}.$$

Definition

For $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $0 < \varepsilon \ll 1$, we have the *Milnor fibration*

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus X \to \mathbb{S}^1.$$

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Example

F is homotopic to a bouquet of $\mu(f)$ spheres. If f is homogeneous, F is diffeomorphic to $f^{-1}(\varepsilon)$, and $H^*(X)$ is concentrated in degree $n-1$ of rank $\mu(f)$.
So the "nearby locus" F stores information about the singularity X.

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- Generalize to motivic zeta function

$$Z_{\mathrm{mot}}(s; f) = \int_{\mathcal{J}_\infty \mathbb{C}^n} \mathbb{L}^{-s \operatorname{ord} f} d\mu$$

- Relation between zeta function $Z(s)$ and the monodromy φ on the Milnor fiber:

$$s_0 \text{ is a pole of } Z(s) \Rightarrow e^{2\pi i s_0} \text{ is an eigenvalue of } \varphi : H^i(F) \to H^i(F)$$

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The (motivic) Monodromy Conjecture [DL98]

We have

$$Z_{\mathrm{mot}}(s) \in K_0(\mathrm{Sch}_{\mathbb{C}})[(1-\mathbb{L}^{-Ns-n})^{-1}]_{(n,N)\in M}$$

such that

$$\exp\left(2\pi i \frac{n}{N}\right) \text{ is an eigenvalue of } \varphi \text{ for } (n,N) \in M$$

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Theorem [DL00]

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. For every $m > 0$,

$$\Lambda(\varphi^m) = \chi(\mathcal{X}_m).$$

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The arc-Floer conjecture [BFdBLN22]

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an isolated singularity at 0. For every $m > 0$, the two spectral sequences are isomorphic, and

$$H_c^{*+(n-1)(2m+1)}(X_m, \mathbb{Z}) \cong HF_*(\varphi^m, +).$$

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  - Components of $\mathcal{X}_m$ correspond to components of $\operatorname{Fix} \varphi^m$.

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Theorem (de Lorenzo Poza, H.)

Let f be an isolated homogeneous singularity in $\mathbb{C}^n$. The arc-Floer conjecture holds.

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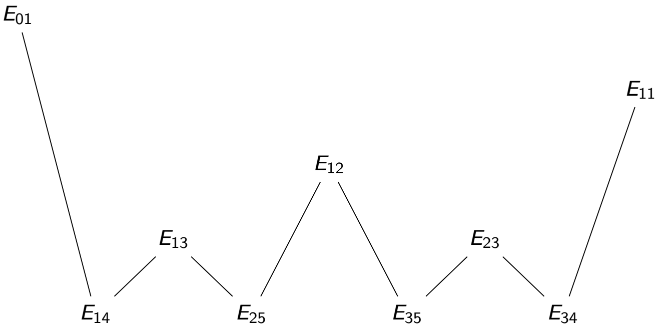
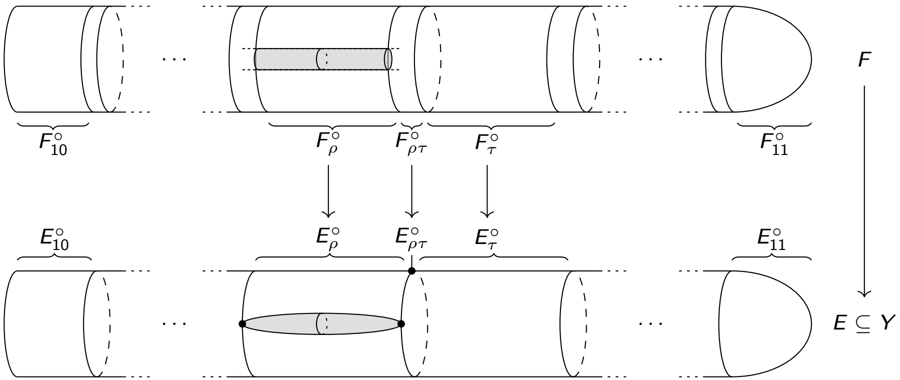
- As of now no conceptual explanation is given for why the conjecture holds.



- Blow up the origin, call the exceptional divisor E_{11} and strict transform $E_{01} = \widetilde{X}$.
- Blow up $E_{01} \cap E_{11}$ and get exceptional divisor E_{12}.
- Repeat until $\rho_1 + \rho_2 + \kappa_1 + \kappa_2 > m$ for any E_ρ adjacent to E_κ.
- This is a subtree of the Stern-Brocot tree with good combinatorial properties.



- Let $Y \to \mathbb{C}^n$ be the resolution of X and E the pre-image of X.
- F can be thought of as an oriented real blowup. Then rounded using the $F^\circ_{\rho\tau}$ part.
- The fixed components of φ^m consists of F_ρ such that $\rho_1 + \rho_2 | m$.

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