Contact Loci of Semihomogeneous Singularities

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Jet and arc spaces

An m**-jet** of a scheme X is a morphism

$$\boxed{\gamma : \underbrace{\operatorname{Spec} k[t]/(t^{m+1})}_{m\text{-disc}} \to X}.$$

- When $m = 0$, a 0-jet is a geometric point of X.
- When $m = 1$, the ring $k[t]/(t^2)$ is the **ring of dual numbers**, so a 1-jet is a tangent vector of X.
- When $m > 1$, a m-jet records higher infinitesimal information of X, up to order m.

An **arc** of a scheme X is a morphism

$$\boxed{\gamma : \underbrace{\operatorname{Spec} k[\![t]\!]}_{\text{disc}} \to X}.$$

- The arc space records infinitesimal information of all orders.

Jet and arc spaces

Fix n. For $X = \mathbb{C}^n$ an m-jet is

$$\gamma : \operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[x_1, \dots, x_n]$$

determined by the images of $x_1, \dots, x_n$ in $k[t]/(t^{m+1})$, so such a morphism corresponds to n terms

$$\boxed{\gamma^i(t) = \gamma_0^i + \gamma_1^i t + \cdots + \gamma_m^i t^m \in \mathbb{C}[t]/(t^{m+1}), \quad i = 1, \dots, n}.$$

The space of m-jets is the m**-th jet space**

$$\boxed{\mathscr{L}_m := \mathscr{L}_m\mathbb{C}^n := \operatorname{Spec} \mathbb{C}[\gamma_j^i]_{j=0,\dots,m}^{i=1,\dots,n}}.$$

The **arc space** of $\mathbb{C}^n$ is the projective limit $\mathscr{L}_\infty = \varprojlim_n \mathscr{L}_m$, parameterizing n-tuples of power series.

Jet and arc spaces

Intuitively, capturing all infinitesimal information of a scheme should be sufficient to retrieve invariants of a scheme even at singular points.

Jet schemes and arc spaces are used in birational geometry and singularity theory mainly through **motivic integration**.

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ cut out a potentially singular hypersurface in $\mathbb{C}^n$. The **restricted contact locus of order** m associated to f at the origin is

$$\boxed{\mathcal{X}_m := \mathcal{X}_m(f, 0) := \{\gamma \in \mathscr{L}_m \mid \gamma(0) = 0, f(\gamma(t)) = t^m (\text{mod } t^{m+1})\}}.$$

Remark The condition $\gamma(0) = 0$ means the jet γ is centered at the origin. We want the contact order $\text{ord}_f \gamma$ to be m, meaning $f(\gamma(t)) = ct^m$ for some $c \neq 0$. Then the locus is *restricted* in the sense that we further want $c = 1$.

Semi - Homogeneous singularities

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d, then

$$\boxed{\text{isolated singularity at } 0 \iff S = \{f = 0\} \text{ is a smooth hypersurface in } \mathbb{P}^{n-1}}$$

$f^{-1}(0) = CS$ 

We study the geometry of the contact loci $\mathcal{X}_m$ for such f, and resolve

- arc-Floer conjecture (ACF),
- embedd~~ing~~ed Nash problem (ENP)

for such singularities (up to a mild constraint on the degree of f).

Previously ACF was only known for plane curves and some special cases of m, and ENP was only known for curves in a surface, hyperplane arrangement, and toric varieties.

Order filtration

The contact locus $\mathcal{X}_m$ admits a natural *increasing* filtration by closed subsets

$$\boxed{F_p\mathcal{X}_m = \{\gamma \in \mathcal{X}_m \mid \operatorname{ord}_t \gamma \geq -p\}}.$$

The *graded pieces* are locally closed subsets given by

$$\boxed{F_{(p)}\mathcal{X}_m = F_p \setminus F_{p-1}}.$$

This filtration induces a spectral sequence

$$\boxed{{}_{\operatorname{ord}}E_1^{p,q} = H_c^{p+q}(F_{(p)}\mathcal{X}_m) \Longrightarrow H_c^{p+q}(\mathcal{X}_m)}.$$

Structural theorem

$\gamma = \gamma_0 + \gamma_1 t + \cdots \gamma_m t^m$

An essential step is to describe the structure of each filtered piece, which allows for computation of the spectral sequence.

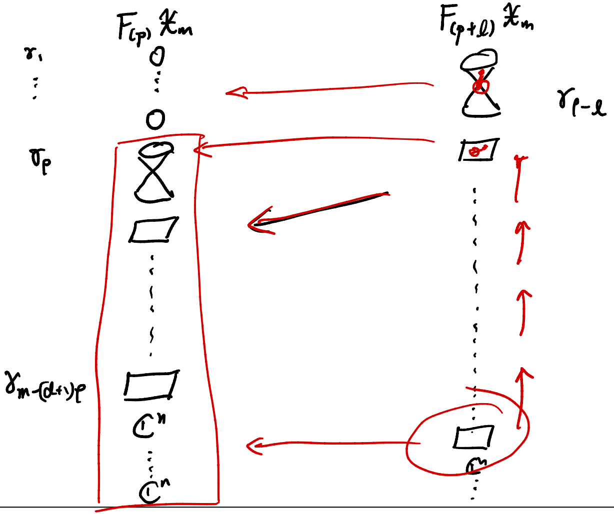
Proposition $1 \leq \rho \leq \frac{m}{d}$

1. $\boxed{F_{(p)}\mathcal{X}_m}$ *is non-empty only for* $-m/d \leq p \leq -1$, *in which case, the following hold. (Write* $\rho = -p$ *for simplicity.)*
2. *If* $\gamma \in F_{(p)}\mathcal{X}_m$, *then the variables* $\boxed{\gamma_0, \ldots, \gamma_{\rho-1}}$ *are zero and* $\gamma_{m-(d-1)\rho+1}, \ldots, \gamma_m$ *are free.*
3. *The variable* γ_ρ *is subject to only one equation, namely*
$$f(\gamma_\rho) = \begin{cases} 0, & \rho \neq m/d; \\ 1, & \rho = m/d. \end{cases}$$

4. *For* $\rho \leq a \leq m-(d-1)\rho$, *denote* $\alpha = a + (d-1)\rho$; *then* γ_a *is cut out by*
$$\boxed{Df(\gamma_\rho)} \cdot \gamma_a + C(\gamma_0, \ldots, \gamma_{a-1}) = \begin{cases} 0, & \alpha \neq m; \\ 1, & \alpha = m, \end{cases}$$
 $\subseteq \mathbb{C}^n$

Structural theorem

Rephrasing of the proposition The piece $F_{(p)}\mathcal{X}_m$ looks like the following:



Structure theorem

The following is the main result that we apply to the Arc-Floer conjecture.

Theorem *The spectral sequence ${}_{\mathrm{ord}}E_1^{p,q}$ degenerates at the first page.*

This means all differentials

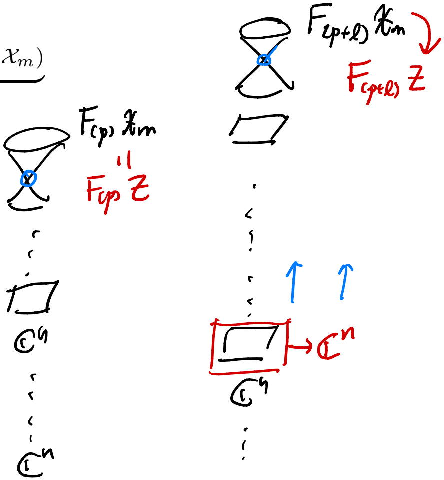
$$d_\ell^{p,q} : \underbrace{H_c^{p+q}(F_{(p)}\mathcal{X}_m)}_{E_\ell^{p,q}} \to \underbrace{H_c^{p+q+1}(F_{(p+\ell)}\mathcal{X}_m)}_{E_\ell^{p+\ell,q-\ell+1}}$$

are zero for $\ell \geq 1$. Taking the dual in the Borel-Moore homology, this map can be thought of as the boundary morphism for cycles, and one intuitively to expect the map to be zero.

Vanishing of differential

Consider an auxiliary space Z which is the same as X, except we remove the hyperplane constraint on the last variable. Let $i : \mathcal{X}_m \hookrightarrow Z$ be the inclusion.

$$\begin{tikzcd}
H_c^{p+q}(F_{(p)}Z) \arrow[r, "d_\ell^{p,q}"] \arrow[d, "i^*"', "\cong"] & H_c^{p+q+1}(F_{(p+\ell)}Z) \arrow[d, "i^*"'] \\
\underbrace{H_c^{p+q}(F_{(p)}\mathcal{X}_m)}_{E_\ell^{p,q}} \arrow[r, "d_\ell^{p,q}"] & \underbrace{H_c^{p+q+1}(F_{(p+\ell)}\mathcal{X}_m)}_{E_\ell^{p+\ell,q-\ell+1}}
\end{tikzcd}$$



Vanishing of differential

It then suffices to show the vanishing of

$$\boxed{i^* : H_c^{p+q+1}(F_{(p+\ell)}Z) \to H_c^{p+q+1}(F_{(p+\ell)}\mathcal{X}_m)}$$

The inclusion $F_{(p+\ell)}\mathcal{X}_m \to F_{(p+\ell)}Z$ is a inclusion of vector bundles given by a natural pullback of the bundle inclusion

$$\boxed{T_{CS^\circ} \to i^*_{CS^\circ}T_{\mathbb{C}^n}}$$

where CS° is the punctured cone

$$CS^\circ = \{f = 0\} \setminus \{0\} \hookrightarrow \mathbb{C}^n.$$

The induced map on cohomology is then the cup product with the euler class of the quotient bundle $e(Q)$. Let $\pi : CS^\circ \to S$ be the projection, then a computation shows

$$\boxed{e(Q) = \pi^*(e(N_{S/\mathbb{P}^{n-1}})) = 0}.$$

$\mathcal{O}_S(d)$

Arc-Floer conjecture

The **Milnor fiber** M_f associated to f is the fiber of the Milnor fibration

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus f^{-1}(0) \to \mathbb{S}^1.$$

$M_f \cong f^{-1}(\delta) \cap B_\varepsilon$

$0 < \delta \ll \varepsilon \ll 1$

which admits a monodromy action $\varphi : M_f \to M_f$.

An application of the m-th contact locus is by Denef and Loeser, stating that

$$\boxed{\chi_c(\boxed{\mathcal{X}_m}) = \Lambda(\varphi^m)}.$$

Seidel noted that the right hand side is the Euelr characteristic $\chi_{\text{Floer}}(\varphi^m)$.

Conjecture (Arc-Fleor conjecture) *We have an isomorphism on the level of cohomology*

$$\mathrm{HF}^*(\varphi^m, +) \cong H_c^{*+(n-1)(2m+1)}(\boxed{\mathcal{X}_m}).$$

Spectral sequences

McLean constructed a Floer theoretic spectral sequence using the action filtration on the Floer complex, giving

$$ {}_{\mathrm{McLean}}E_1^{p,q} \Longrightarrow \mathrm{HF}^{p+q}(\varphi^m, +). $$

Budur, Fernández de Bobadilla, Lê, and Nguyen constructed another spectral sequence on the contact locus,

$$ {}_{\mathrm{BFLN}}E_1^{p,q} \Longrightarrow H_c^{p+q}(\mathcal{X}_m). $$

The two spectral sequences are shown to be isomorphic on the first page, further hinting the conjecture.

In particular,

$$ \boxed{\text{Both sequences degenerate at first page} \Longrightarrow \text{(AFC) is true.}} $$

Our result on ACF

We have shown an equivalence of spectral sequences

$$ {}_{\mathrm{ord}}E_\ell^{p,q} \cong {}_{\mathrm{BFLN}}E_\ell^{p,q} $$

and its degeneration on the first page.

We further show the degeneration of ${}_{\mathrm{McLean}}E_1^{p,q}$ under some assumptions about the degree d of f, and conclude

Theorem *Let* $f : (\mathbb{C}^n, 0) > (\mathbb{C}, 0)$ *be a semihomogeneous germ of degree* d *in* $n \geq 3$ *variables. If*

$$ d < n/2 \quad \text{or} \quad 2n - 2 < d, $$

then (AFC) holds.

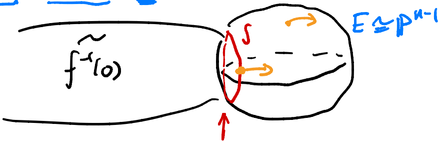
$\left[\frac{n}{2}, 2n-2\right]$

Resolution of homogeneous singularities

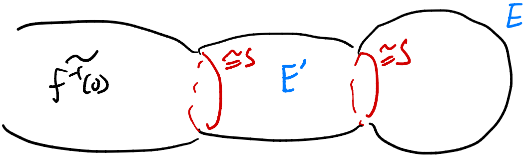
A (semi)homogeneous isolated singularity is resolved after one blow-up at the origin, with the exceptional divisor $S = \{f = 0\} \subseteq \mathbb{P}^{n-1}$ such that

$$\boxed{N_E} = \operatorname{ord}_E f = d.$$

$Bl_0 \mathbb{C}^n$



We may further blow up S, and obtain a new divisor E', then $N_{E'} = d + 1$.

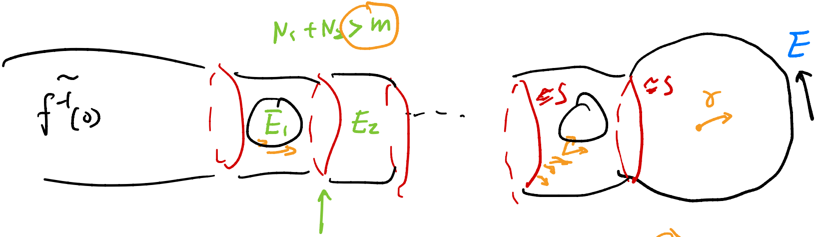


m-separating resolution

Suppose $\mu : X \to (\mathbb{C}^n, Y, 0)$ is a log resolution of the singularity Y, with exceptional divisor $E = \sum N_i E_i$ that is SNC.

> **Definition** μ is ***m*-separating** if for any $i \neq j$ such that $E_i \cap E_j \neq \phi$, we have
> $$N_i + N_j > m.$$

After repeatedly blowing up intersections, we can make a log resolution m-separating. For homogeneous singularities, we get a chain:



$$\underline{m} = \mathrm{ord}_f \gamma = \mathrm{ord}_E \gamma \cdot N_E$$

Embedded Nash problem

Now suppose $\mu : X \to (\mathbb{C}^n, Y, 0)$ is a m-separating log resolution. The contact locus decomposes

$$\mathcal{X}_{m,E} = \{\gamma \in \mathcal{X}_m \mid \tilde{\gamma}(0) \in E\}$$

where E is an exceptional component, and $\tilde{\gamma}$ is the lift of γ by evaluative criterion. This is only non-empty if $N_E|m$, (we call that an m-**divisor**) because the contact order of the lift has to be m/N_E for γ to contact f at order m.

Say $\mathcal{E}$ is the set of exceptional components (plus the strict transform), then

$$\boxed{\mathcal{X}_m = \bigsqcup_{\substack{E \in \mathcal{E} \\ N_E|m}} \mathcal{X}_{m,E}}$$

$\overline{\mathcal{X}_{m,E}} \supseteq \mathcal{X}_{m,F}$

This is set theoretically disjoint, but not topologically, i.e. the closure of some components might intersect others.

Embedded Nash problem

For every irreducible component C of $\mathcal{X}_m$, there is an exceptional divisor E such that

$$C = \overline{\mathcal{X}_{m,E}}.$$

The Embedded Nash problem asks

$$\boxed{\text{For which } E \in \mathcal{E} \text{ is } \overline{\mathcal{X}_{m,E}} \text{ an irreducible component?}}$$

We provide a complete answer for semi-homogeneous isolated singularities.

m-valuations

The Nash problem can be solved by describing valuations ord_E corresponding to each $E \in \mathcal{E}$.

Definition

1. A divisorial valuation v on Y is an **m-valuation** if $v = \mathrm{ord}_E$ for some divisor $E \in \mathcal{E}$ such that $N_E | m$.
2. v is a **dlt** m**-valuation** if $v = \mathrm{ord}_E$ such that E appears in a dlt modification of Y.
3. v is a **contact** m**-valuation** if $v = \mathrm{ord}_E$ such that $\boxed{\overline{\mathcal{X}_{m,E}}}$ is an irreducible component of $\mathcal{X}_m$.
4. v is an $\boxed{\textbf{essential}}$ m**-valuation** if $v = \mathrm{ord}_E$ such that E appears in every m-separated log resolution of Y.

Proposition

$$\{dlt\} \subseteq \{contact\} \subseteq \{essential\}$$

$$\boxed{\begin{array}{llll} \phi \subseteq & \{*\} \subseteq & \{\text{every}\} & d<n \\ \{\text{every}\} \subseteq & \{\text{every}\} \subseteq & \{\text{every}\} & d\geq n \end{array}}$$

Our results describing valuations

Let Y be a semi-homogeneous isolated singularity of degree d in $\mathbb{C}^n$.

Theorem (Answer to ENP)

1. *If $d < n$, then the only contact valuation is the exceptional divisor obtained from blowing up the origin.*
2. *If $d \geq n$, every m-divisor of a m-separating resolution gives a contact valuation.*

Theorem

1. *If $d < n$, then are no dlt valuations.*
2. *If $d \geq n$, every m-divisor of a m-separating resolution gives a dlt valuation.*

Theorem

Every m-divisor of a m-separating resolution gives an essential valuation.

$d=2$ $m = 8$

Stern Brocot tree.

$$1 \; 8 \; 7 \; 6 \; 5 \; 4 \; 7 \; 3 \; 8 \; 5 \; 7 \; 2$$

$E_0 = f^{-1}(0)$ E_2 E_3 E_1

$N_E \mid m$

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