You have heard of jets which are like higher order tangent vectors.

A tangent vector takes a function and returns the derivative of that function along the tangent direction.

A n-jet takes a function and returns the order n-Taylor polynomial along its "direction".

Let's make everything algebraic:

Differential geometry	Algebraic geometry
Tangent vector	tangent vector
$v \in T_x X$, $v: C^\infty(X) \to \mathbb{R}$.	$v \in \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[t]/(t^2), X)$ $v(0) = x$.
n-jet	n-jet
$v: C^\infty(X) \to \mathbb{R}[t]/(t^{n+1})$	$v \in \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[t]/(t^{n+1}), X)$.

$\mathrm{Spec}\,\mathbb{C}[\varepsilon]/\varepsilon^2$ is the ring of dual number with a closed pt and 1 infinitesimal direction

$\mathrm{Spec}\,\mathbb{C}[t]/t^{n+1}$ is the disk of order n.

<u>Defn</u> The <u>m-th jet scheme</u> of a variety m is the space of m-jets.

$\leftrightsquigarrow$ the scheme representing the functor:

$$(\mathrm{Aff}_k)^{op} \longrightarrow \mathrm{Set}$$
$$\mathrm{Spec}\,R \longmapsto \mathrm{Hom}_{\mathrm{Sch}_k}(\mathrm{Spec}\,R[t]/(t^{n+1}), X).$$

$\leftrightsquigarrow$ the Weil restriction

$$\mathrm{Res}_{(k[t]/t^{n+1})/k}(X \otimes k[t]/t^{n+1}).$$

$m = \infty \Rightarrow$ arc space. projective limit.

Why care about the jet space? Suppose X is smooth of dim d.

$\mathcal{L}_0 X = X$. $\mathcal{L}_n X \to \mathcal{L}_m X$ is affine with fiber $\mathbb{A}^{(n-m)d}$

so $H^*(\mathcal{L}_m X) = H^*(X)$.

So what?

The power of jet space is seen when given a map.

If $f: X \to Y$ induce $f_n : \mathcal{L}_n(X) \to \mathcal{L}_n(Y)$.

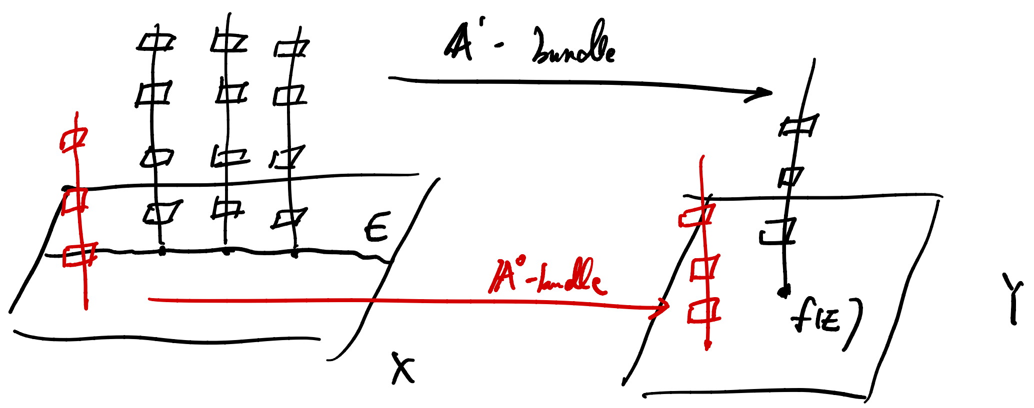
If f birational, then f_∞ injective on $\mathcal{L}_\infty(X) \setminus \mathcal{L}_\infty(E)$.

If f proper then bijective on $\mathcal{L}_\infty(X) \setminus \mathcal{L}_\infty(E) \to \mathcal{L}_\infty(Y) \setminus \mathcal{L}_\infty(f(E))$

At level 0, $f: X \to Y$ is a generic map. does not relate $H^*(X)$ to $H^*(Y)$.

At level ∞. f_∞ a bijection, does not say anything about the cohomology.

At level n, cohomology data can be retained as f_n is a "piecewise-bundle"



Cohomology: It is not completely correct to say f_n detects cohomology.

the notion we care about in motivic integration is Euler characteristics.

Recall, if X variety and have stratification $X = \coprod X_i$

then $\chi_c(X) = \sum \chi_c(X_i)$

More generally, we care about the Grothendieck ring

$$K_0(Var) = \{\{X\} \mid X \text{ variety}\}/\sim$$

where $\{X\} = \{Y\} + \{X \setminus Y\}$ for $Y \subseteq X$ closed.

$\{X \times Y\} = \{X\} \cdot \{Y\}$.

also can write $\{X\} = M(X) = e(X)$, the motive of X.

Then $\chi_c : K_0(Var) \to \mathbb{Z}$ is a ring homomorphism.

so to find $\chi_c(X)$, we can express $\{X\}$ as
a sum of easier terms in $K_0(\mathrm{Var})$, then take χ_c.

Ring multiplicative identity $\{pt\} = \mathbb{L}^0$
additive identity $\{\emptyset\}$
set $\{\mathbb{A}^n\} = \mathbb{L}^n$.

<u>Ex</u> $\{\mathbb{P}^n\} = \overbrace{\{\mathbb{A}^n\} + \{\mathbb{A}^{n-1}\} + \dots + \{\mathbb{A}^1\} + \{pt\}}^{\text{stratification.}}$

we know $\chi_c(\mathbb{L}^n) = (-1)^n$, so $\chi_c(\mathbb{P}^n) = \sum_{i=1}^{n} (-1)^i$

If $X \to Y$ has fiber $\mathbb{A}^e$ then $\{X\} = \{Y\} \cdot \mathbb{L}^e$.

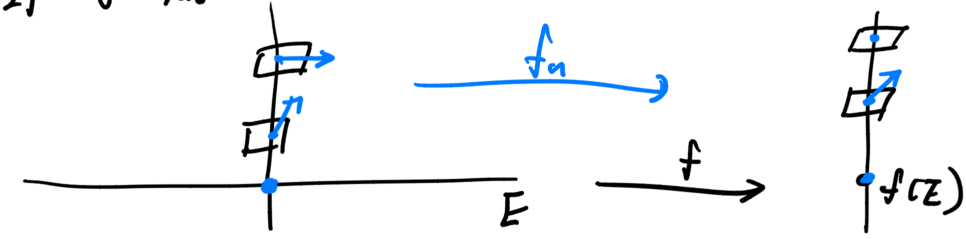
There are other invariants compatible with stratifications
such as HD: $K_0(\mathrm{Var}) \to \mathbb{Z}[u,v]$
MHS: $K_0(\mathrm{Var}) \to K_0(\mathrm{MHS})$.

Suppose $f: X \to Y$ birational with exceptional divisor E.
If I know about Y, and want χ_c of X,
then I can take $\{X\} = \{X \setminus E\} + \{E\}$
$= \{Y \setminus f(E)\} + \{E\}$
but how does E relate to anything on Y?

I can try $\{X\} = \{\mathcal{L}_n(X)\} \cdot \mathbb{L}^{-nd}$
then compare $\mathcal{L}_n(X)$ to $\mathcal{L}_n(Y)$.

Know $X \setminus E \xrightarrow{\sim} Y \setminus f(E)$, so $\{\mathcal{L}_n(X \setminus E)\} = \{\mathcal{L}_n(Y \setminus f(E))\}$
but same problem. How does
$\{\mathcal{L}_n(X) \setminus \mathcal{L}_n(X \setminus E)\}$ compare to $\{\mathcal{L}_n(Y) \setminus \mathcal{L}_n(Y \setminus f(E))\}$?

If γ has center in E



Then some directions are collapsed.

So one expects f_n to be a vector bundle.

How big are the fibers?

Since jets record derivatives, fibers are measured by Jacobians.

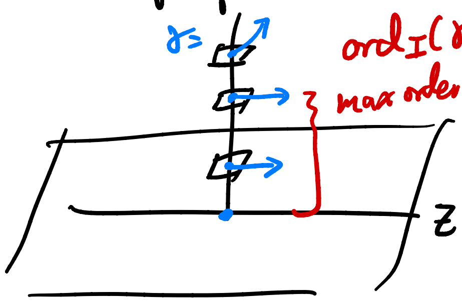
The Jacobian of f is $\mathrm{Jac}_f = K_X - f^* K_Y = E$

The Jacobian order is

$$\mathrm{ord\,Jac} : \mathcal{L}_n(X) \to \mathbb{N}$$

$$\gamma \mapsto \mathrm{valuation}(\gamma^{-1}\mathrm{Jac})$$

<u>Ex</u> I ideal sheaf of $Z \subseteq X$



$\mathrm{ord}_I(\gamma) =$ max order of directions contained in Z.

Idea: If γ has a high degree of coincidence with E, then it gets collapsed more by f.

If $\gamma \in \mathcal{L}_\infty(E)$ then $\mathrm{ord\,Jac}\,\gamma = \infty$.

<u>Thm</u> ord Jac gives decomposition

$$\mathcal{L}_n(X) = \underbrace{\coprod_{2e \le n} \mathrm{ord\,Jac}^{-1}(e) \;\coprod}_{\text{stratification}} \Big(\bigcup_{2e > n} \mathrm{ord\,Jac}^{-1}(e) \Big)$$

<u>Thm</u> ($\mathbb{A}^e$-fibration lemma) for $2e \le n$, $\mathrm{ord\,Jac}^{-1}(e) \longrightarrow f_n(\mathrm{ord\,Jac}^{-1}(e))$ has fiber $\mathbb{A}^e$.

Hence $\{\mathcal{L}_n(X)\}\cdot\mathbb{L}^{-nd} = \sum_{e\le n}\{\operatorname{ord}\mathrm{Jac}^{-1}(e)\}\cdot\mathbb{L}^{-nd} + \sum_{e\ge n}\{\operatorname{ord}\mathrm{Jac}^{-1}(e)\}\cdot\mathbb{L}^{-nd}$

$$= \sum_{e\le n}\{f_n\operatorname{ord}\mathrm{Jac}^{-1}(e)\}\cdot\mathbb{L}^{e}\cdot\mathbb{L}^{-nd} + \cdots$$

Thm for $e\le n<n'$, $\operatorname{ord}\mathrm{Jac}_{n'}^{-1}(e) = (\pi^{n'}_{n})^{-1}\operatorname{ord}\mathrm{Jac}_n^{-1}(e)$ (cylinder)

$$\Rightarrow \{\operatorname{ord}\mathrm{Jac}_{n'}^{-1}(e)\} = \{\operatorname{ord}\mathrm{Jac}_n^{-1}(e)\}\cdot\mathbb{L}^{(n'-n)d}.$$

$\Rightarrow$ well defined motivic measure

$$\mu_X(\operatorname{ord}\mathrm{Jac}^{-1}(e)) := \{\operatorname{ord}\mathrm{Jac}_n^{-1}(e)\}\mathbb{L}^{-nd}.$$

<u>Conclusion</u> $\{X\} = \sum_{e\in\mathbb{N}}\mu(\operatorname{ord}\mathrm{Jac}^{-1}(e))\cdot\mathbb{L}^{e} + \underbrace{\mu(\operatorname{ord}\mathrm{Jac}^{-1}(\infty))}_{=\mu(\mathcal{L}_\infty(E))}$

$=0$

$\mu(\mathcal{L}_\infty(E))=0$ since $\mathcal{L}_n(E)$ has $\dim = n\dim E$

but the shift is $-nd$

so as $n\to\infty$, the limit vanishes.

(to be precise, we work in completion $\widehat{K_0(\mathrm{Var})[\mathbb{L}^{-1}]}$).

Thm $f_n\operatorname{ord}\mathrm{Jac}^{-1}(e)$ also gives stratification on $\mathcal{L}_\infty(Y)$. so

$$\int_{\mathcal{L}_\infty(X)}\mathbb{L}^{-\operatorname{ord}\mathrm{Jac}}\,d\mu_X = \int_{\mathcal{L}_\infty(Y)}d\mu_Y$$

More generally. a cylinder $C\subseteq\mathcal{L}_\infty(X)$ is of form $C=(\pi^{\infty}_{n})^{-1}(C_n)$

for some n, $C_n\subseteq\mathcal{L}_n(X)$ constructible.

$$\mu_X(C) = \{C_n\}\mathbb{L}^{-n\cdot d}.$$

$h : \mathcal{L}_\infty(X) \to \mathbb{Z}$ is measurable if

$h^{-1}(e)$ cylinder, so

$$\int_{\mathcal{L}_\infty(X)} \mathbb{L}^h \, d\mu_X = \sum_e \mu_X(h^{-1}(e)) \cdot \mathbb{L}^e .$$

If $X \to Y$ crepant, then $\mathrm{ord}\,\mathrm{Jac} = 0$, so $\{X\} = \{Y\}$

$\chi_c(X) = \chi_c(Y)$.

Generalizations: If X singular, then arcs can be weird,

the bad part (arcs near singularity)

can be recorded using differentials on X.

$\{X\}$ does not give the correct χ_c. Let X be r-Gorenstein

Then $\{X\}_{stringy} := \int_{\mathcal{L}_\infty(X)} d\mu_X^{Gor}$

$$:= \int_{\mathcal{L}_\infty(X)} \mathbb{L}^{\frac{1}{r}\mathrm{ord}\, I} \, d\mu_X,$$

where $\mathrm{Im}\left((\Omega_X^d)^r \to \omega_X^{[r]} \right) = I\, \omega_X^{[r]}$, $\omega_X^{[r]} = (\omega_X^{r})^{vv}$.

gets rid of bad things on singular locus.

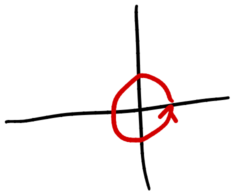
Then if $X \to Y$ crepant resolution X sm, Y r-Gor,

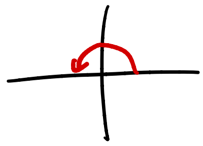
$\{X\} = \{Y\}_{st}$ by change of variable formula

Twisted arcs

$X=\mathbb{C} \to Y=\mathbb{C}$

Consider the 2-covering $z \mapsto z^2$





A loop on X corresponds to half a loop of Y.

So if we take the collection of all loops on X,

then that corresponds to loops and half loops on Y.

Now let $Y = V/G$ quotient singularity.

$\mathcal{X} = [V/G]$ crepant resolution by stack.

An arc on Y does not lift to arc on $\mathcal{X}$, but "half arc" on $\mathcal{X}$

These are called twisted arcs on $\mathcal{X}$.

Let $\mathcal{X}$ be a stack. A twisted jet on $\mathcal{X}$ of degree ℓ is

$$\mathcal{D} = \left[\operatorname{Spec} k[\![t^{1/\ell}]\!]/(t^{n+1}) \big/ \mu_\ell\right] \longrightarrow \mathcal{X}$$

Let $J_n^\ell \mathcal{X}$ be the n-th twisted jet space of degree ℓ.

If working over formal disk, one can define $\mathrm{Utg}\, \mathcal{X}$

s.t. $\mathcal{L}_n \mathrm{Utg}(\mathcal{X}) = \coprod_\ell J_n^\ell \mathcal{X}$.

So there is ordinary motivic integration on $\mathrm{Utg}(\mathcal{X})$.

Have maps

$$\begin{array}{ccc} & & \mathrm{Utg}(\mathcal{X}) \\ & & \downarrow \pi^{utg} \\ \mathcal{X} & \xrightarrow[\text{cms}]{\pi} & X \end{array}$$

$$\int_{\mathcal{L}_\infty \mathrm{Utg}(\mathcal{X})} \mathbb{L}^{-j_{\pi^{utg}}} \, d\mu_{\mathrm{Utg}(\mathcal{X})} = \int_{\mathcal{L}(X)} d\mu_X^{Gor}$$

Can define j_π and $j_{\pi^{utg}}$ to be ord Jac.

The difference $j_\pi - j_\pi^{\text{wdg}}$ is the shift fun $s_{\mathcal{X}}$

$$\text{So } \int_{J_\infty(\mathcal{X})} \mathbb{L}^{-j_\pi + s_{\mathcal{X}}} d\nu_{\mathcal{X}} = \int_{\mathcal{L}_\infty(X)} d\mu_X^{\text{Cor}}.$$

Although LHS is not really $\{\mathcal{X}\}$ in any sense,

it is a class $\{\mathcal{X}\}_{\text{orb}}$

$$\{\mathcal{X}\}_{\text{orb}} = \sum_{\mathcal{Z} \subseteq I\mathcal{X}} \{\mathcal{Z}\} \cdot \mathbb{L}^{\underbrace{-s_{\mathcal{X}}(\mathcal{Z})}_{\text{fractional shift.}}}$$

$$H^*_{\text{orb}}(\mathcal{X}) := \bigoplus_{\mathcal{Z}} H_c^{*-2s(\mathcal{Z})}(\mathcal{Z}) \otimes \underbrace{\mathbb{Q}(-s(\mathcal{Z}))}_{\text{fractional Tate shift}}$$

is the orbifold cohomology

we say the orbifold df we say the orbifold cohomology is a cohomological realization of $HD(Y)$.

In general, $$\int_{J_\infty \mathcal{X}} \mathbb{L}^{s_{\mathcal{X}} - j_f} d\nu_{\mathcal{X}} = \int_{J_\infty Y} \mathbb{L}^{s_Y} d\nu_Y.$$

<u>Artin stacks</u> If Y has log-terminal singularity, then $\exists$ crepant resolution

$\mathcal{X} \to Y$, $\mathcal{X}$ Artin stack.

Then arcs on Y lift to warps on $\mathcal{X}$

$\mathcal{W}(\mathcal{X})$ warping stack, a T-point is

$$\begin{array}{l} \mathcal{J} \xrightarrow{\text{rep}} \mathcal{X} \\ \downarrow \text{flat, f.p., gms, coh } \Delta. \\ T \end{array}$$

and Jac is generalized to "height function" $ht_{\mathcal{X}/Y}$.

$$\int_{\mathcal{L}_\infty(\mathcal{Y})} d\mu_{\mathcal{Y}} = \int_{C_{\mathcal{X}}} \mathbb{L}^{-\mathrm{ht}_{W(\mathcal{X})/\mathcal{Y}}} \, d\mu_{W(\mathcal{X})}.$$

$C_{\mathcal{X}} \subseteq |W(\mathcal{X})|$ is a canonical locus of good maps that lift arcs of $\mathcal{Y}$.

not $d\mu^{\mathrm{Gor}}_{\mathcal{Y}}$ as $\mathrm{ht}_{W(\mathcal{X})/\mathcal{Y}}$ contains data about singularity of $\mathcal{Y}$.

Alternatively, can define weight (generalizing shifts)

$$\int_{\mathcal{L}_\infty(\mathcal{Y})} d\mu^{\mathrm{Gor}}_{\mathcal{Y}} = \int_{A_{\mathcal{X}}} \mathbb{L}^{-\mathrm{wt}_{\mathcal{X}}} \, d\nu_{\mathcal{X}}$$

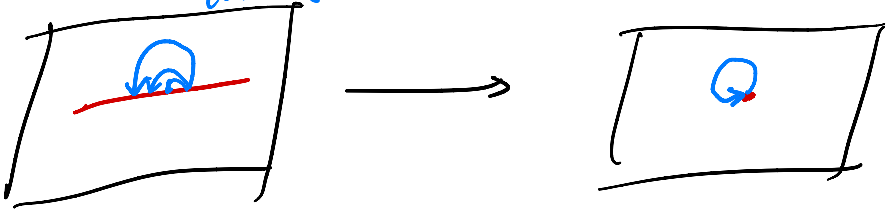
where $A_{\mathcal{X}} \subseteq |\mathcal{J}_\infty(\mathcal{X})|$ is a locus of "good lifts".

<u>Question</u>: is there a cohomological interpretation of

$\mathrm{HD}(\mathrm{RHS}) = \mathrm{HD}(\{\mathcal{Y}\}_{st})$ like orbifold cohomology?

Consider Quotient with high dimensional orbit.

lots of lifts.



Why stress the condition wt finite values?

maybe $wt_{x'/x}$ infinite, giving infinite HD.

What are $f_\infty^{-1}(wt^{-1}(e))$? cylinder?

components?

$A_x \cap wt^{-1}(e)$

$wt^{-1}(e)$ cylinder by 4.21

locally constant

$f_\infty^{-1}(A_x \cap wt^{-1}(e))$ cylinder?

connected comps?

Compactify curve in $\mathbb{P}^2$

A line in $\mathbb{P}^2$ can be given by lines

passing 0 in $Z=0$.

lines on $\mathbb{P}^2_{\mathbb{C}} \longleftrightarrow$ planes in $\mathbb{C}^3$ passing 0

$\mathbb{P}^2_{\mathbb{C}}$ on $Z \neq 0$ has coordinates $[\frac{X}{Z} : \frac{Y}{Z} : 1]$

$= (x, y) \cong \mathbb{C}^2$.

$Z = 0$ has $[X : Y : 0] \cong \mathbb{P}^1$.

$\downarrow$ $Y \neq 0$ $[\frac{X}{Y} : 1 : 0]$

$= (x) \cong \mathbb{C}^1$

$Y = 0$ has $[1:0:0] = \mathbb{C}^0$.

$= \{\infty\}$.

In $Z \neq 0$, have curve $x^2+y^2=1$

so extends to all of $\mathbb{P}^2_{\mathbb{C}}$ by $X^2+Y^2=Z^2$

Conversely a polynomial in X, Y, Z

makes sense as a fcn on $\mathbb{P}^2$

if it is homogeneous.

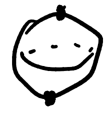
Points added are at $Z=0$, $\cong \mathbb{P}^1$

so $X^2+Y^2=0$ in $\mathbb{P}^1$

$= [1 : \pm i]$

call these two points ∞_1, ∞_2. then

get $X \cup \{\infty_{1,2}\}$

$=$ 

Ex find a biholomorphic map

$$\mathbb{P}^1 \longrightarrow \{x^2+y^2=z^2\} \subseteq \mathbb{P}^2.$$

$$[s:t] \longmapsto [\,?\,,\,?\,,\,?\,]$$

first find $\mathbb{P}^1 \to \{xy=z^2\}$

then find $\{xy=z^2\} \cong \{x^2+y^2=z^2\}$.

Ex compactify $y^2 = x(x-1)(x+1)$.