NON-REDUCTIVE GIT, K-STABILITY, AND STACKS

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ABSTRACT. This is the notes taken from talks given by Joshua Jackson, Jacopo Stoppa, and David Rydh at the Equivariant Methods in Geometry workshop of Cambridge university in January 2024.

CONTENTS

1. REDUCTIVE AND NON-REDUCTIVE GEOMETRIC INVARIANT THEORY

Let k be an algebraically closed field of characteristic 0 (so that reductive and linearly reductive coincide). Suppose G acts on a scheme X. The quotient space X/G is in general not a scheme, and the method of GIT follows the strategy of throwing away bad orbits and gluing together some orbits. Locally, if G acts on $\operatorname{Spec} A$, then we would want the quotient to be

$$\varphi : \operatorname{Spec} A \to \operatorname{Spec} A^G.$$

In general, $\varphi : X \to W$ is a *good quotient* if

(1) φ is G-invariant, surjective and affine.
(2) $\mathcal{O}_W \cong \varphi_* \mathcal{O}_X^G$, meaning locally the quotient is a map to the invariants, of form $\operatorname{Spec} A \to \operatorname{Spec} A^G$.
(3) If W_1, W_2 are distinct, closed and G-invariant, then so are $\varphi(W_1)$ and $\varphi(W_2)$. This means we only glue minimally.

Note that good quotients are categorical, and when all orbits are closed, the quotient W agrees with the naive quotient X/G. In the case $W = X/G$, we say φ is a *geometric quotient*.

Example 1.1. An example of a quotient which is spec of invariants, but is not good, is $\mathbb{G}_a$ on the affine variety $X = \mathrm{SL}(2)$ by $\alpha = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ acting on $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2)$ by multiplication. In this case we have $A^G = k[c, d]$, so the quotient is $X \to \mathbb{A}^2$, but the image does not contain the origin, so it is not surjective. Furthermore, there also exists a $\mathbb{G}_a$-action on $\mathbb{A}^3$ where A^G does not separate closed orbits. These are problems that occour when G is non-reductive.

Definition 1.2. A linear algebraic group is an affine group scheme. It is *reductive* if every finite dimensional representation V is completely reducible, meaning that there exists a decomposition $V = \oplus V_i$ where V_i is irreducible. It is *unipotent* if it is isomorphic to a closed subgroup of the group of upper triangular matrices with 1 in the diagonal.

Every linear algebraic group H has a decomposition $H = U \rtimes R$ with U unipotent, called its unipotent radical, and R reductive.

Example 1.3. For a filtration $0 \subsetneq V_0 \subsetneq ... \subsetneq V_m = V$, let $P \subseteq \mathrm{GL}(n)$ be the parabolic subgroup preserving each V_i, so P is block upper triangular. The unipotent radical of P is the block upper triangular with identities on the diagonal blocks, and the reductive part are the block diagonal matrices.

1.1. **Mumford's GIT for reductive groups.** We first recall the usual GIT on reductive groups.

Theorem 1.4 (Affine reductive GIT)**.** *The affine GIT quotient for reductive G acting on $X = \operatorname{Spec} A$ is given by*

$$\varphi : X \to \operatorname{Spec} A^G =: X /\!\!/ G.$$

To generalize to projective GIT, knowing good quotients are local on target, we need to find an invariant affine covering, and then glue using affine GIT. For this approach, we use the following notion.

Definition 1.5. A *linearization* of an action G on X is an equivariant embedding $X \hookrightarrow \mathbb{P}^n$ such that G acts on $\mathbb{P}^n = \mathbb{P}(V)$ for some representation V of dimension $n + 1$.

Remark 1.6. If X is normal and L is ample, then there exists linearization $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*\mathcal{O}(1) = L^n$ for some $n > 0$.

Theorem 1.7 (Projective reductive GIT)**.** *Let G act on $X = \operatorname{Proj} A$ linearly. Then $A^G \hookrightarrow A$ determines a rational map $\varphi : X \dashrightarrow X /\!\!/ G =: \operatorname{Proj} A^G$, where φ is defined on the semi-stable locus*

$$X^{ss} = \{x \in X | \text{ there exists } f \in A^G \text{ such that } f(x) \neq 0\}.$$

The map $\varphi : X^{ss} \to X /\!\!/ G$ is a good quotient. When restricted to the stable locus

$$X^s = \{x \in X^{ss} : \ Gx \text{ is closed in } X^{ss} \text{ and } \mathrm{stab}_G(x) \text{ is finite}\},$$

the quotient is a geometric quotient $\varphi : X^s \to X^s/G$. The quotient $X /\!\!/ G$ is projective and X^s/G is quasi-projective.

A classical application of GIT is on the space $W \cong \mathbb{P}(k[x_0, \ldots, x_n]_d^*)$ of hypersurfaces in $\mathbb{P}^n$ of degree d. The group $G = \mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}(n + 1)$ acts on W. Except for some low n, d, the moduli space of smooth hypersurfaces is given by W^{sm}/G, and such surfaces are stable. Thus $W /\!\!/ G$ is a reasonable compactification of W^{sm}/G. It is often difficult to compute invariants directly, so to find out what is added in this compactification, the Hilbert-Mumford criteria is commonly used.

When T is a torus acting on $X \hookrightarrow \mathbb{P}(V)$, decompose $V = \oplus V_{\alpha_i}$ where $\alpha_i \in \operatorname{Lie} T^*$ are the weights of the representation. For $x = [x_0 : \cdots : x_n]$, define

$$\mathrm{Conv}(x) = \text{convex hull}\{\alpha_i | x_i \neq 0\} \subseteq \operatorname{Lie} T^*.$$

Then the Hilbert-Mumford criterion states $x \in X^{(s)s}$ if and only if 0 is in (the interior of) Conv(x). More generally, we have the following.

Theorem 1.8 (Hilbert-Mumford criterion)**.** *Let G be reductive acting on X linearly. Let $T \leq G$ be a maximal torus in G. Then $x \in X^{(s)s}$ if and only if $0 \in Conv^{(\circ)}(gx)$ for all $g \in G$.*

Equivalently we may phrase this using 1-parameter subgroups $\lambda : \mathbb{G}_m \to G$. Set $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$ and $\mu(x, \lambda)$ to be minus the weight of λ on x_0. Then $x \in X^{(s)s}$ if and only if $\mu(\geq)0$ for all such λ.

Exercise 1.9. Consider the natural action of SL(2) on $\mathbb{P}^1$. Let $X = (\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^N$ via the Veronese embedding. This is a linearization. Show that a point $p = (p_1, \dots p_n)$ is (semi)-stable if and only if $(\leq)\frac{n}{2}$ of the coordinates coincide.

Example 1.10. The moduli of stable curves (with nodes) of a fixed genus can be obtained as a GIT quotient of some Chow scheme (or Hilbert scheme). The moduli space of semi-stable coherent sheaves on a projective scheme X can be constructed as a GIT quotient of some Quot scheme.

1.2. **Stratification of unstable locus.** To study the unstable objects, we would like a way to measure instability. Considering the Hilbert-Mumford criterion, we could take the distance of the origin from Conv(gx) for $g \in G$, or find a λ such that $\mu(x, \lambda)$ is minimal. However, $\mu(x, \lambda)$ is not bounded as we can reparametrize λ, thus we defined the following notion. Let $\|\cdot\|$ be a positive definite, Weyl invariant (with respect to a maximal subgroup T of G), integral valued bilinear form on

$$\chi_*(T) = \{\lambda : \mathbb{G}_m \to T\}.$$

Example 1.11. When $G = \mathrm{SL}(n+1)$ we can take $\|\cdot\|$ to be the Killing form.

Being Weyl invariant means that this is a well defined norm on $\chi_*(G)$. The *Normalized Hilbert Mumford function* is $M(x, \lambda) := \mu(x, \lambda)/\|\lambda\|$ and

$$\Lambda_x := \{\lambda | M(x, \lambda) \text{ is minimal}\}.$$

Theorem 1.12. *For a fixed unstable point $x \in X - X^{ss}$, we have the following.*

(1) There exists some λ achieving $\inf_\lambda M(x, \lambda)$, so $\Lambda_x \neq \phi$.
(2) There exists a parabolic subgroup $P(x) \subseteq G$ such that $P(x) = P(\lambda)$ for all $\lambda \in \Lambda_x$.Here $P(\lambda)$ is defined by

$$\{g \in G | \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\}.$$

(3) Any $\lambda_1, \lambda_2 \in \Lambda_x$ are conjugate by and element in $P(x)$.

Example 1.13. When $G = \mathrm{SL}(n + 1)$, $P(\lambda)$ is the parabolic subgroup preserving the weight filtration of λ.

This way we have associated to a point x a worst λ. When $G = T$, let $\beta(x)$ be the closest point to 0 on Conv(x). Then we take λ_β to be such a λ and set $P_\beta = P(\lambda_\beta)$. We have the following loci

$$Y_\beta^{ss} = \{x \in X : \beta(x) = \beta\}, \quad Z_\beta^{ss}\{x \in X^{\lambda_\beta} : \beta(x) = \beta\} \subseteq Y$$

and projection map

$$p_\beta : Y_\beta^{ss} \to Z_\beta^{ss}, \quad x \mapsto \lim \lambda_\beta x.$$

Theorem 1.14 (Hesselink-Kempf-Kirwan-Ness (HKKN) stratification)**.** *Let G be a reductive group with a choice of $\|\cdot\|$ acting on $X \subseteq \mathbb{P}^n$ linearly. Then there exists a G-invariant locally closed stratification*

$$X = \bigsqcup_{\beta \in B} S_\beta$$

indexed over a finite set $B \subseteq \mathrm{Lie}\, T^*$, such that

(1) $S_0 = X^{ss}$.
(2) $S_\beta = G\cdot Y_\beta^{ss} \cong G\times_{P_\beta} Y_\beta^{ss} =: G\times Y_\beta^{ss}/P_\beta$, *where* P_β *acts on* $G\times Y_\beta^{ss}$ *by* $n\cdot(g,x) = (gn^{-1}, nx)$.
(3) $\overline{S}_\beta \subseteq \cup_{\|\gamma\|\geq\|\beta\|} S_\gamma$.

Now we may consider the quotient by G even in the unstable locus. Property (2) implies that quotienting S_β by G is the same as quotienting Y_β^{ss} by P_β, which is helpful as P^β has more characters. But the problem is that P_β is non-reductive, this motivates the study of non-reductive GIT.

Example 1.15. Consider the moduli of n points on $\mathbb{P}^1$ with the action of SL(2) as before. We have

$$X^{ss} = \{p \in X|\text{at most } n/2 \text{ of the points coincide}\},$$
$$B = \{n, n-2, n-4\dots\} \subseteq \mathbb{Z}_+,$$
$$Y_\beta^{ss} = \{p \in X|\beta \text{ points coincide at } \infty \in \mathbb{P}^1\},$$
$$Z_\beta^{ss} = \{p \in X|\beta \text{ points coincide at } \infty\text{, the rest coincide at } 0\},$$
$$S_\beta = \{p \in X|\text{exactly } \beta \text{ points coincide}\}.$$

Example 1.16. For $m \gg n \gg 0$, the HKKN stratification for the moduli of coherent sheaves on Y is given by Harder-Narasimhan types. It is conjectured that the stratification for moduli of curves stabilize for $n \gg 0$ in a similar manner.

1.3. Non-reductive GIT.

Given $a \in \mathbb{N}^{n+1}$, the weighted projective space is

$$\mathbb{P}(a) = \mathbb{A}^{n+1} - \{0\}/\mathbb{G}_m$$

where the $\mathbb{G}_m$ action has weight a. A hypersurface in $\mathbb{P}(a)$ of degree d is cut out by a homogeneous $f \in k[x_0, \dots, x_n]$ of degree d, where the degrees on x_i are given by a. There are parameterized by

$$\mathrm{Hyp}_d(\mathbb{P}(n)) = \mathbb{P}(k[x_0, \dots, x_n]_d^* \cong \mathbb{P}^N.$$

The moduli space is then the quotient by the automorphism group, but this group is not reductive.

Example 1.17. The space $\mathbb{P}_{[x,y,z]}(1,1,2)$ has automorphisms $z \mapsto z + ax^2 + bxy + cy^2$ for $a, b, c \in \mathbb{G}_a^3$, and $\mathbb{G}_a^3$ is a normal subgroup in Aut (it is in fact its unipotent radical).

We recall the properties of reductive GIT and outline the problems for non-reductive groups. Let G be reductive acting on X.

(1) For $X = \operatorname{Spec} A$, $\varphi : X \to X /\!\!/ G = \operatorname{Spec} A^G$ is a good quotient and A finitely generated implies A^G finitely generated.
(2) For $X \subseteq \mathbb{P}^n$ projective and G acting linearly, $\varphi : X^{ss} \to X /\!\!/ G = \operatorname{Proj} A^G$ is a projective good quotient. The stable locus $X^s \to X^s/G$ gives a quasi-projective geometric quotient. The (semi)-stable locus can be computed using Hilbert-Mumford criterion.

Now let H be non-reductive.

(1) (Nagata) A^H might not be finitely generated.
(2) $\operatorname{Spec} A \to \operatorname{Spec} A^H$ might not be surjective and its image not necessarily a scheme.
(3) $\operatorname{Spec} A^H$ might not separate orbits.
(4) There might be no non-trivial maps $G_m \to H$, so we do not have Hilbert-Mumford criterion.

If H acts on A and U is a normal subgroup, then $R \cong H/U$ acts on A^U. So let us consider the case when our non-reductive group is unipotent.

Theorem 1.18. *If U is unipotent, acting on a quasi-affine X, then all orbits are closed.*

Corollary 1.19. *If $X \to Y$ is a good U-quotient, then it is geometric.*

This is very restrictive as we now need a constancy condition on $\dim \mathrm{stab}_U(x)$ for all $x \in X$, otherwise the quotient would not be geometric. Let us consider the case when the action is free.

Definition 1.20. A linear algebraic group $H = U \rtimes R$ has *graded unipotent radical* if there exists $\lambda : \mathbb{G}_m \to Z(R)$ such that the adjoint representation of $\lambda\mathbb{G}_m$ acting on $\operatorname{Lie} U$ has strictly positive weights.

Remark 1.21. If H is a subgroup of $\mathrm{GL}(n)$ such that matrices in U are upper triangular, and $\lambda(\mathbb{G}_m)$ consists of diagonal matrices with strictly decreasing weights, then λ grades U.

Example 1.22. In HKKN stratification, λ_β grades the unipotent radical U_β of P_β. The automorphism group $\mathrm{Aut}(\mathbb{P}(a))$ also has graded unipotent radical.

1.4. **The $\hat{U}$ theorem.** Let $H \cong U \rtimes R$ have graded unipotent radical by λ. Suppose H acts on X linearly by $\rho : H \to \mathrm{GL}(V)$, where $X \hookrightarrow \mathbb{P}^n$ is given by the choice of a very ample bundle. Write $V_{\min} \subseteq V$ the minimal $\lambda(\mathbb{G}_m)$ weight space. Define

$$Z_{\min} = \mathbb{P}(V_{\min}) \cap X$$

which is a closed projective subvariety of X. Consider the open subset

$$X^\circ_{\min} = \{x \in X | \lim_{t\to 0} \lambda x \in Z_{\min}\}$$

with projection $p_\lambda : X^\circ_{\min} \to Z_{\min}$ given by $x \mapsto \lim \lambda x$. Let

$$\hat{U} := U \rtimes_\lambda \mathbb{G}_m \leq H.$$

Theorem 1.23 (Bérczi, Doran, Hawes, Kirwan 2016)**.** *Assume* $\mathrm{stab}_U(z) = \{e\}$ *for all* $z \in Z_{\min}$. *Then*

1. *There exists projective geometric quotient* $X^{s,\hat{U}} := X^\circ_{\min} - UZ_{\min} \to X /\!\!/ \hat{U}$, *where* $X /\!\!/ \hat{U}$ *is the Proj of the invariants of some "well-adapted" linearization.*
2. *The usual GIT quotient by R gives a good quotient*

$$X^{ss,H} \to (X /\!\!/ \hat{U}) /\!\!/ (R/\lambda) =: X /\!\!/ H$$

where $X /\!\!/ H$ is also the Proj of some invariants.

Remark 1.24. In the above setting, we also have a non-reudctive version of Hilbert-Mumford criterion.

The version of the proof we present uses *twisted affine GIT* as follows. Suppose G is reductive, acting on $X = \operatorname{Spec} A$ with character $\chi : G \to \mathbb{G}_m$. Say $f \in A$ is a χ^n invariant, meaning $gf = \chi^n(g)f$. Let A^{χ^n} be the semi-invariants with respect to χ^n. Then we have a graded ring $\oplus_n A^{\chi^n} \cong A[w]^G$ where G acts on w^n by χ^n. The twisted affine quotient is the birational map $X \dashrightarrow X /\!\!/_\chi G =: \operatorname{Proj} \oplus A^{\chi^n}$ induced by $A[w]^G \hookrightarrow A[w]$, which is projective over $\operatorname{Spec} A^G$. The domain of this map is

$$X^{\chi-ss} = \{x \in X | \exists n > 0, f \in A^{\chi^n}, f(x) \neq 0\}$$

which can be computed by a variant of the Hilbert-Mumford criterion.

Every unipotent group has a sequence of normal subgroups

$$\{e\} \triangleleft U^1 \triangleleft \cdots \triangleleft U^\ell = U, \quad U^i/U^{i-1} \cong \mathbb{G}_a.$$

Thus we focus on the quotient of a $\mathbb{G}_a$ action on $X = \operatorname{Spec} A$. This is the same as a co-action

$$\sigma^* : A \to A \otimes k[t].$$

Definition 1.25. A *derivation* $D : A \to A$ is a k-linear map satisfying the Leibniz rule. It is *locally nilpotent* if there is some $f \in A$ such that $D^n f = 0$ for some n.

Proposition 1.26. *There exists a bijective correspondence between*

$$\{\mathbb{G}_a\text{-actions on } A\} \leftrightarrow \{\text{locally nilpotent derivations}\}$$
$$\left(\sigma^* : f \mapsto \sum \frac{D^n(f)}{n!} t^n\right) \leftrightarrow D$$
$$\sigma^* \to \leftrightarrow \left(D : f \mapsto \frac{\partial}{\partial t}|_{t=0}\sigma^*(f)\right)$$
$$A^{\mathbb{G}_a} \leftrightarrow \ker D.$$

Example 1.27. The action $\mathbb{G}_a$ on $\mathbb{A}^3$ given by $x \mapsto x + ay + a^2z/2, y \mapsto y + az$, $z \mapsto z$ corresponds to the derivation $D(x) = y, D(y) = z, D(z) = 0$.

Note that if $f \in \ker D = A^{\mathbb{G}_a}$, then we have induced map

$$D_f : A_f \to A_f \quad D(g/f^n) = D(g)/f^n.$$

Given $I \subseteq A$ with $D(I) \subseteq I$, we also have $D : A/I \to A/I$. One can check that $x \in X^{\mathbb{G}_a}$ if and only if $D(A) \subseteq \mathfrak{m}_x$.

Definition 1.28. A *slice* for the action $\mathbb{G}_a$ on X is an element $s \in A$ with $D(s) = 1$.

The reason for the name can be seen in the following proposition.

Proposition 1.29. *The following are equivalent.*

(1) There exists a slice $s \in A$.
(2) $A \cong A^{\mathbb{G}_a}[s]$ and $D = \frac{d}{ds}$.
(3) There exists an equivariant isomorphism $X \cong X' \times \mathbb{A}^1$ for some affine X'.

If there exists a slice, then by (2), the ring $A \cong A^{\mathbb{G}_a}[s]$ is graded with $A_n := s^{-n}A^{\mathbb{G}_a}$ for $n < 0$. So the $\mathbb{G}_a$ action extends to a $\mathbb{G}_a \rtimes \mathbb{G}_m$ action where $\mathbb{G}_m$ grades $\mathbb{G}_a$ in the sense of graded unipotent radical. Since all weights are negative, $\lim_{t\to 0} \lambda(t)x$ exists for all $x \in X$.

Definition 1.30. A *local slice* is an element $s \in \ker D^2 - \ker D$. If s is a local slice with $D(s) = t \neq 0$, we get a slice $s/t \in A_t$ for the induced map $D_t : A_t \to A_t$. So we can construct locally trivial quotient away from the *plinth ideal*

$$pl(A) = \ker D \cap \operatorname{im} D.$$

Suppose $\mathbb{G}_a \rtimes \mathbb{G}_m$ acts on X with $\mathbb{G}_m$ acting on $\operatorname{Lie}\mathbb{G}_a$ with weight $\ell > 0$, then we have a grading $A \cong \oplus A_n$ and $D : A_n \to A_{n+\ell}$.

Proposition 1.31. *Suppose* $\lim \lambda x$ *exists for all x, then there exists a slice if and only if $X^{G_a} = \phi$.*

Proposition 1.32. *Let $\mathbb{G}_a$ act on X. Then there exists a slice if and only if the action extends to $\mathbb{G}_a \rtimes \mathbb{G}_m$ such that all limits exists and* $\operatorname{stab}_{\mathbb{G}_a}(z) = \{e\}$ *for all $z \in Z = \{\lim \lambda x\}$.*

Sketch of proof of the $\hat{U}$ theorem. We have $p : X^\circ_{\min} \to Z_{\min} = \mathbb{P}(V_{\min}) \cap X$. The proof follows the following steps.

1. Work affine locally on $Z = \operatorname{Spec} A_0 \subseteq Z_{\min}$. We have $p^{-1}(z) = \operatorname{Spec} A =: Y$ as p is affine, and $A = \oplus_{m\leq 0} A_m$, graded by λ.
2. By construction $\mathbb{G}_m$ acts on Y and $Z = \{\lim \lambda y\}$.
3. Choose $\{e\} \neq U^1 \triangleleft \cdots \triangleleft U^\ell$ with $U^i/U^{i-1} = \mathbb{G}_a$.
4. Consider $\hat{U}^1 = \mathbb{G}_a \rtimes \mathbb{G}_m$ acting on Y, so it satisfies the above proposition, which gives us a slice $s_1 \in A$ and $A \cong A^{\mathbb{G}_a}[s]$ is finitely generated. Therefore $\operatorname{Spec} A^{\hat{U}^1} = Y/\mathbb{G}_a$. Now U^2/U^1 acts on $Y/\mathbb{G}_a$, and by induction, A^U is finitely generated.

(5) Consider the twisted affine quotient by $\hat{U}$

$$\operatorname{Proj} A[w]^{\hat{U}}$$

where $\hat{U}$ acts on w by the character $\chi : \hat{U} \to \hat{U}/U \to \mathbb{G}_m$ corresponding to λ. Note that unipotent groups do not have any characters.

(6) As A_0 is U-invariant by $D : A_0 \to A_{0+\ell} = 0$, one can show $A_0 = A^{\hat{U}}$.

(7) The weighted quotient is projective over Z, so they glue to something projective over $Z_{\min}$, and thus projective over k.

(8) One can show $Y^{\chi-ss} = Y - UZ$, which gives $X^{\circ}_{\min} - UZ_{\min}$ after gluing.

(9) Obtain a projective geometric quotient for $X^{\circ}_{\min} - UZ_{\min}$.

□

1.5. **Applications.**

(1) Using non-reductive GIT (repeatedly), we may take quotients of the HKKN strata to classify unstable objects. For example, one can obtain the moduli spaces of unstable sheaves on projective schemes.
(2) We can obtain the moduli of hypersurfaces of the weighted projective space.
(3) Enumerative geometry on Hilbert schemes of $\mathbb{C}^n$.

2. K-stability and infinite dimensional moment maps

Let M be a compact complex manifold with an almost complex structure J. Say J is integrable if $[T^{[0,1]}M, T^{[0,1]}M] \subseteq T^{0,1}$. Consider the problem of constructing the space $\mathfrak{M} = \mathfrak{M}(M_{C^\infty})$ parametrizing complex structures on M, such that the points are complex structures up to isomorphisms and a family of complex structures would induce a map to $\mathfrak{M}$. Such a space $\mathfrak{M}$ is in general not Hausdorff as J may jump.

Example 2.1. Let $M = \mathbb{P}^2\#(\#_m\overline{\mathbb{P}^2})$ where $\overline{\mathbb{P}^2}$ is taken with opposite orientation. For $m > 4$, M has a continuous family of distinct complex structures such that $M \cong \operatorname{Bl}_{p_1,\dots,p_m}\mathbb{P}^2$. Choose λ a one-parameter subgroup of PGL(3), then as we move $p_1, \dots, p_m$ using λ onto a line, the complex structure jumps.

A Riemannian metric g on (M, J) is *Hermitian* if J is a g-isometry. It is *Kähler* if $\nabla_g J \equiv 0$ for the Levi-Civita connection of g. In this case, we get a symplectic form $\omega = g(J\cdot, \cdot)$ which corresponds to a class $[\omega] \in H^2_{\mathrm{dR}}(M, \mathbb{R}) \cap H^{1,1}_{\bar\partial}(M)$. Call $(M, J, [\omega])$ a polarized complex manifold.

The Ricci curvature $\operatorname{Ric}(g)$ is determined by a 2-form $\operatorname{Ric}(\omega)$ called the Ricci form. Similarly we have the scalar curvature form $\operatorname{Scal}(\omega)$. Using local holomorphic coordinates one can show $d\operatorname{Ric}(\omega) = 0$, and $[\operatorname{Ric}(\omega)] \in H^2_{\mathrm{dR}}(M, \mathbb{R}) \cap H^{1,1}_{\bar\partial}(M)$ is independent of the class $[\omega]$.

Theorem 2.2 (Schumacher)**.** *There exists a Hausdorff complex space $\mathfrak{M}$ parametrizing manifolds $(M, J, [\omega])$ such that*

$$[\omega] = -[\operatorname{Ric}(\omega)] = -c_1(M) = c_1(K_M).$$

Such spaces are called canonically polarized spaces.

Remark 2.3. The same holds for Calabi-Yau spaces and for those spaces where $[\operatorname{Scal}(\omega)] = c$ for constant c is solvable for some ω.

2.1. **Donaldson-Fujiki moment map.** Let M be compact, ω_0 a Kähler form. The space

$$\mathscr{J}_{\omega_0} = \mathscr{J} = \{\omega_0\text{-compatible } J\}$$

is an infinite dimensional pre-Hilbert manifold. It has a tautological almost Kähler structure $\mathbb{J}$ with a symplectic form given by $\Omega_J(A,B) = \int_M \operatorname{tr}(JAB)\frac{\omega_0^n}{n!}$. The group of exact symplectomorphisms

$$\mathscr{G} = \operatorname{Ham}(\omega_0) := \{\text{time 1 flow of time dependent Hamiltonian vector fields of } \omega_0\}$$

acts on $\mathcal{J}$ by pullback. Our goal is to perform GIT on this space.

Theorem 2.4 (Fujiki, Donaldson)**.** *The action is a Hamiltonian action with respect to* Ω *with a unique moment map given by the Hermitian scalar curvature, up to a constant shift.*

On the integrable locus $\mathscr{J}^{int} \subseteq \mathscr{J}$ cut out by the condition $[T^{[0,1]}, T^{[0,1]}] \subseteq T^{[0,1]}$, the moment map is given by

$$s(\omega_0, J) - \hat{s} := \operatorname{Scal}(\omega_0(\cdot, J\cdot)) - \hat{s}$$

where $\hat{s}$ is a constant. We have $\operatorname{Lie}\mathscr{G} = C_0^\infty(M,\omega_0)$. Identify $\operatorname{Lie}\mathscr{G}^\vee$ with $\operatorname{Lie}\mathscr{J}$ using the L^2 inner product $\langle\cdot,\cdot\rangle$, so the moment map is a map $\mu : \mathscr{J} \to \operatorname{Lie}\mathscr{G}$. The moment map condition is

$$\partial_{t=0}\langle s(\omega_0, J_t) - \hat{s}, \varphi\rangle = -\Omega(\mathcal{L}_{X_\varphi^{\omega_0}} J, J)$$

for any path J_t with $J_0 = J$ inside $\mathscr{J}^{int}$.

Even though we can not defined a complexification of $\mathscr{G}$, the orbits of a $\mathscr{G}^{\mathbb{C}}$-action on $\mathscr{J}$ are well-defined. For $f \in \mathscr{G}$, let $\mathcal{O} = \mathscr{G}\cdot J$, we have

$$T_{f^*J}\mathcal{O} = \{\mathcal{L}_{X_\varphi^{\omega_0}} f^*J : \varphi \in C_0^\infty(M,\omega_0)\}.$$

The complexified distribution is

$$\mathcal{D} = T_{f^*J}\mathcal{O} \oplus \mathbb{J}T_{f^*J}\mathcal{O}.$$

Proposition 2.5. *The distribution* $\mathcal{D}$ *is formally integrable, namely* $[\mathcal{D},\mathcal{D}] \subseteq \mathcal{D}$.

For simplicity, we consider the special case $[\omega_0] = c_1(L)$ for some positive line bundle $L \to M$. Then we can describe the integral submanifolds of $\mathcal{D}$ using the following

$$\mathcal{F} = \{(\omega \in [\omega_0], f \in \operatorname{Diff}_0(L \to M) | f^*\omega = \omega_0)\}.$$

For any fixed J, we obtain a map

$$\Phi : \mathcal{F} \to \mathscr{J}$$

by $(\omega, f) \mapsto f^*J$.

Proposition 2.6. $\Phi(\mathcal{F})$ *is an integral submanifold of* $\mathcal{D}$ *through the point* J.

Thus we morally have obtained a "complexification" of $\mathscr{G}$ acting on $\mathscr{J}$. For $f \in \operatorname{Diff}_0(L \to M)$, note that $s(\omega_0, f^*J) = f^*s((f^{-1})^*\omega_0, J)$. Thus solving for $\mu = 0$ is the same as for solving $s(J) = \hat{s}$ in the $\mathscr{G}^{\mathbb{C}}$ orbvits, which is the same as solving $s(\omega) = \hat{s}$ for $\omega \in [\omega_0]$.

2.2. **K-Stability.** Let us try to make sense of the Hilbert-Mumford criterion for the $\mathscr{G}^{\mathbb{C}}$-action. Let $\mathfrak{X} \to \mathbb{C}$ be a family of complex manifolds and a $\lambda = \mathbb{C}^*$-action. If a $\mathbb{C}^*$-action on $\mathfrak{X}_0 = M_0$ is generated by $X \in K := \mathscr{G} \cap \operatorname{Aut}(M_0, J_0)$ and ϕ the Hamiltonian of X , then $\mu(J_0, \lambda) = \int_{M_0} \phi(s(\omega_0, J_0) - \hat{s})\frac{\omega_0^n}{n!}$ is the *Futaki invariant*.

A flat family $\pi : \mathfrak{X} \to \mathbb{C}$ is a test configuration if $\mathfrak{X}$ is normal with relatively ample line bundle $\mathcal{L} \to \mathfrak{X}$ which is $\mathbb{C}^*$-equivariant, such that the generic fiber $(\mathfrak{X}_1, \mathcal{L}_1)$ is isomorphic to (M, L). Gluing using $\mathbb{C}^*$ action, we compactify to $\overline{\mathfrak{X}} \to \mathbb{P}^1$. Define

$$DF(\mathfrak{X},\mathcal{L}) := \frac{n\mu}{n+1}\mathcal{L}^{n+1} + K_{\mathfrak{X}/\mathbb{P}^1}\cdot\mathcal{L}^n$$

where $\mu = \frac{c_1(M)\cdot\mathcal{L}^{n-1}}{\mathcal{L}^n}$. It is conjectured that $s(\omega) = \hat{s}$ is solvable if and only if $DF(\mathfrak{X}, \mathcal{L}) \geq 0$ with equality only if $\mathfrak{X}_0 \cong M$. THis is called K-poly-stability, and is proven for the Fano case by Chen-Donaldson-Sun.

Suppose now $\omega_0 = c_1(L)$, $s(\omega_0, J) = \hat{s}$. The group of deformations of J is $K = \mathscr{G} \cap \operatorname{Aut}(M, J)$, which is compact, and we get a reductive group $K^{\mathbb{C}} = \operatorname{Aut}(M, L)$. Thus $K^{\mathbb{C}}$ acts on $\tilde{H}^1 \subseteq H^1(TM)$, the first order deformations preserving L.

Theorem 2.7. *The small deformations J' for which*

$$s(\omega, J') = \hat{s}$$

is solvable correspond to $K^{\mathbb{C}}$-poly-stable orbits of the $K^{\mathbb{C}}$-action on $\tilde{H}^1$.

Corollary 2.8. *A small deformation (M', L') of (M, L) is solvable if and only if it is K-poly-stable.*

Theorem 2.9 (Arezzo-Pacard-Singer)**.** *Suppose (M, ω, J) is cscK and $s(\omega, J) = \hat{s}$. Fix $p_1, \ldots, p_m \in M$ and moment map $\mu : M \to \operatorname{Lie} K^\vee$. If $\sum \mu(p_i) = 0$ and $\operatorname{Aut}(\operatorname{Bl}_{p_1,\ldots,p_m} M) = \{e\}$, then*

$$(\operatorname{Bl}_{p_1,\ldots,p_m} M, [\pi^*\omega - \varepsilon^2 \sum E_i])$$

is still cscK for $\varepsilon \ll 1$ where E_i are the exceptional divisors.

3. Algebraic stacks and moduli spaces

A scheme is a sheaf $X : \mathrm{AffSch}^{op} \to \mathcal{S}et$ such that there exists a Zariski atlas $X = \cup \operatorname{Spec} A_i$. An algebraic space is as above, but with étale atlas $\sqcup \operatorname{Spec} A_i \to X$ which is étale surjective.

Definition 3.1. An algebraic stack is a sheaf

$$\mathfrak{X} : \mathrm{AffSch}^{op} \to \mathcal{G}pd$$

with smooth atlas $\sqcup \operatorname{Spec} A_i \to X$ which are smooth and surjective.

Let $\mathcal{M}_g(T)$ be families of smooth projective genus g curves over T, so it is a groupoid whose objects are curves $C \to T$ and whose morphisms are isomorphisms relative to T. Then there exists an atlas that makes $\mathcal{M}_g$ an algebraic stack.

There exists a topological space $|\mathfrak{X}| = \{\operatorname{Spec} k \to \mathfrak{X}\}/\sim$. For any $x : \operatorname{Spec} k \to \mathfrak{X}$, the group $\operatorname{Aut}(x)$ is a group scheme over k.

Definition 3.2. $\mathfrak{X}$ is Deligne-Mumford if for all $x \in |\mathfrak{X}|$, $\operatorname{Aut}(x)$ is an étale group scheme (in characteristic 0, this means they are finite groups). In this case smooth atlas are equivalent to étale atlas.

3.1. Quotient stacks.

Let G act on a scheme X, then we have an algebraic stack $\mathfrak{X} = [X/G]$ with atlas $X \to [X/G]$ which is a G-torsor. We have the following correspondences.

Quotient stack $[X/G]$	G-equivariant geometry of X
$\lvert\mathfrak{X}\rvert$	G-orbits of X
x	$p^{-1}(x)$
$p(x)$	Gx
$\operatorname{Aut}(p(x))$	$\operatorname{stab}(x)$
$\operatorname{QCoh}(\mathfrak{X})$	$\operatorname{QCoh}^G(X)$
$\mathcal{F}$	$p^*\mathcal{F}$
$\Gamma(\mathfrak{X}, \mathcal{F})$	$\Gamma(X, p^*\mathcal{F})^G$

Remark 3.3. The stack $[X/G]$ does not remember the group G. Writing $\mathfrak{X} = [X/G]$ is the same as giving a representable map $\mathfrak{X} \to [*/G] = BG$.

Example 3.4. $\mathfrak{X} = BG$ has space $|\mathfrak{X}| = *$ and $\mathrm{Aut}(*) = G$. It is given by

$$\mathfrak{X}(T) = \{G\text{-torsor } E \to T\}$$

Example 3.5. $[\mathbb{A}^{n+1} - 0/\mathbb{G}_m] = \mathbb{P}^n$.

Example 3.6. Consider $\mathbb{G}_m$-action on $\mathbb{A}^n - 0$ with weights $d_1, \dots d_n$. The quotient is the weighted projective stack $\mathcal{P}(d_1, \dots, d_n)$ which has finite automorphisms.

Example 3.7. The stack $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ has two orbits. $\{x = 0\}$ with stabilizer $\mathbb{G}_m$ and $\{x \neq 0\}$ with stabilizer $\{e\}$. We write

$$|\Theta| = \cdot_1 \rightsquigarrow \cdot_0.$$

3.2. **Moduli spaces.**

Definition 3.8. A *coarse moduli space* to $\mathfrak{X}$ is $\pi : \mathfrak{X} \to \mathbb{X}$ where $\mathbb{X}$ is an algebraic space such that

(1) π is a universal homeomorphism,
(2) π is initial/categorical in the sense that any other $\mathfrak{X} \to \mathbb{Y}$ factors through $\mathbb{X}$ uniquely.

Remark 3.9. If $\mathfrak{X} = [X/G]$ and $\pi : \mathfrak{X} \to \mathbb{X}$ is a coarse moduli space, then $X \to \mathbb{X}$ is a geometric quotient.

Example 3.10. $\mathcal{M}_g \to M_g$, $BG \to *$ and $\mathcal{P}(d_i) \to \mathbb{P}(d_i)$ are all coarse moduli spaces. If G is reductive acting on X projective, then $[X^s/G] \to X^s/G$ is a quasi-projective coarse moduli space.

Definition 3.11. A *good moduli space* to $\mathfrak{X}$ is $\pi : \mathfrak{X} \to \mathbb{X}$ such that

(1) π is quasi-compact and quasi-separated,
(2) $\pi_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathbb{X})$ is exact,
(3) $\mathcal{O}_{\mathbb{X}} \cong \pi_* \mathcal{O}_{\mathfrak{X}}$.

Example 3.12. Consider $BG \to *$. It is quasi-separated if and only if G is a finite type group scheme. $\pi_* : V \mapsto V^G$ is exact when G is (linearly reductive). We have $\pi_*(k) = k^G = k$ is always true.

Example 3.13 (Affine GIT)**.** Suppose G is reductive acting on $X = \operatorname{Spec} A$, then $\mathfrak{X} = [X/G] \to X /\!\!/ G$ is a good moduli space because $\pi_* : \mathrm{Mod}^G(A) \to \mathrm{Mod}(A^G)$ is exact, and $\pi_*(A) = A^G$.

Example 3.14 (Projective GIT)**.** $[X^{ss}/G] \to X /\!\!/ G$ is a good moduli space.

3.2.1. *Topological properties of good moduli spaces.*

(1) π is universally closed.
(2) For all $x \in \mathbb{X}$, there exists a unique closed point in $x_0 \in \pi^{-1}(x)$ (analogous to orbit separating for good quotients).

Example 3.15. Consider $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ with weight $1, -1$. The orbits in $\mathbb{A}^2$ are cut out by $\{xy = t\}$ where the x-axis, y-axis specialize to the origin. $\mathfrak{X}$ is not separated as we can find sections $\mathbb{X} \to \mathfrak{X}$ by $s(t) = (1, t)$ or $s(t) = (t, 1)$.

3.2.2. *Categorical properties.* Any $\mathfrak{X} \to \mathbb{Y}$, factors through π uniquely.

3.2.3. *Finiteness (Hilbert's 14th problem).* If $\mathfrak{X}$ is of finite type over an Noetherian S, then so is $\mathbb{X}$, and π_* preserves coherence.

3.2.4. *Base change.* For any $\mathbb{X}' \to \mathbb{X}$, $\mathfrak{X} \times_{\mathbb{X}} \mathbb{X}' \to \mathbb{X}'$ is a good moduli space.

3.2.5. *Luna fundamental lemma.* If we have

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \downarrow{\scriptstyle gms} & & \downarrow{\scriptstyle gms} \\ \mathbb{X} & \xrightarrow{g} & \mathbb{Y} \end{array}$$

with f étale and $x \in |\mathfrak{X}|$ closed, $f(x) \in \mathfrak{Y}$ is closed in the fiber over $\mathbb{Y}$, and $\operatorname{stab} x \cong \operatorname{stab} f(x)$, then

- there exists an open neighbourhood $U \subseteq \mathbb{X}$ of $\pi(x)$ over which g is étale and the diagram is Cartisian,
- F takes closed points to closed points and $\operatorname{stab} x' = \operatorname{stab} f(x')$ for all x' in a neighbourhood of x.

Example 3.16. A non-example is when $\mathfrak{X} = \mathbb{X} = \mathbb{P}^1$ and $\mathfrak{Y} = [\mathbb{A}^n/\mathbb{G}_m]$. Then f does not map closed point to closed point.

Example 3.17. Another non-example is when $\mathfrak{X} = \mathbb{X} = *$ and $\mathfrak{Y} = B\mathbb{Z}_2$.

Remark 3.18. There are other notions such as adaquate moduli space, which cover GIT for reductive but not necessarily linearly reductive groups. This is a special case of topological moduli space which satisfies all above properties except the base change property only holds for flat base change.

3.3. **Local structure theorem.** The local structure theorem states that good moduli spaces are étale locally affine GIT.

Theorem 3.19 (Alper, Hall, Rydh)**.** *Let $\mathfrak{X}$ be an algebraic stack of finite presentation over k and $x \in \mathfrak{X}(k)$ such that*

(1) all automorphism groups are affine,
(2) $G_x = \operatorname{Aut}(x)$ is linearly reductive,

then there exists $\mathfrak{W} = [\operatorname{Spec} A/G_x]$ such that

$$\begin{array}{ccc} (\mathfrak{W}, w) & \xrightarrow{\text{étale}} & (\mathfrak{X}, x) \\ \downarrow{\scriptstyle gms} & & \\ \operatorname{Spec} A^{G_x} & & \end{array}$$

where w is a closed point mapped to x, and $\operatorname{stab} w = \operatorname{stab} x$.

Remark 3.20. $\mathfrak{X}$ does not remember G, be does remember G_x, so localy structure is a canonical quotient around x. If $\mathfrak{X} = [X/G]$ for X affine, then one can find $\operatorname{Spec} A \hookrightarrow X$ where $G_x \leq G$ acts on A.

Corollary 3.21. *Suppose $\mathfrak{X} \to \mathbb{X}$ is a good moduli space and $x \in |\mathbb{X}|$. Let $x_0 \in |\mathfrak{X}|$ be the unique closed point in $\pi^{-1}(x)$. Then we further have*

$$\begin{array}{ccc} [\operatorname{Spec} A/G_x] = (\mathfrak{W}, w) & \xrightarrow{\text{étale}} & (\mathfrak{X}, x) \\ \downarrow{\scriptstyle gms} & & \downarrow{\scriptstyle gms} \\ \operatorname{Spec} A^{G_x} = \mathbb{W} & \xrightarrow[\text{étale}]{} & \mathbb{X} \end{array}$$

3.4. **Existence theorem for good moduli space.**

Theorem 3.22 (Keel-Mori)**.** *Let $\mathfrak{X}$ be an algebraic stack with finite automorphism groups. The following are equivalent.*

(1) There exists coarse moduli space $\mathfrak{X} \to \mathbb{X}$ where π is separated.
(2) The inertia stack is finite (this should be thought of as a weak separation condition).

Remark 3.23. If $\mathfrak{X}$ is separated, then the inertia $I_{\mathfrak{X}} \to \mathfrak{X}$ is separated.

Example 3.24. A non-example to the theorem is the quotient stack of the line with doubled origin, by $\mathbb{Z}_2$ which swaps the origin and act trivially everywhere else. $I_{\mathfrak{X}} \to \mathfrak{X}$ is not finite, and we shall see it is also not S-complete.

Theorem 3.25 (Alper, Halpern-Leistner, Heinloth)**.** *Let $\mathfrak{X}$ be an algebraic stack of finite type with affine diagonal, then $\mathfrak{X}$ admits a separated good moduli space if and only if $\mathfrak{X}$ is Θ-complete and S-complete.*

Remark 3.26. If $\mathfrak{X}$ has finite stabilizer then the above theorem recovers Keel-Mori. In this case, Θ-completeness is trivial, and S-completeness is equivalent to separatedness.

Definition 3.27. For a discrete valuation ring R, let $\Theta_R = \Theta \times \operatorname{Spec} R$. $\mathfrak{X} \to \mathfrak{Y}$ is Θ-*complete* if it satisfies the following (dimension 2 version of) valuative criterion.

$$\begin{array}{ccc} \Theta_R - 0 & \longrightarrow & \mathfrak{X} \\ \big\downarrow & \overset{\exists !}{\nearrow} & \big\downarrow \\ \Theta_R & \longrightarrow & \mathfrak{Y} \end{array}$$

Example 3.28. $\mathfrak{X} = [\mathbb{P}^1/\mathbb{G}_m]$ has three orbits, with 1 specializing to ∞ and 0, is not Θ-complete.

Definition 3.29. Let $\overline{ST} = [\mathbb{A}^2/\mathbb{G}_m]$ with weights $1, -1$ which has $\mathbb{A}^1$ as a good moduli space. Say $\mathfrak{X} \to \mathfrak{Y}$ is S-complete if it satisfies the valuative criterion for $\overline{ST}_R$.

Sketch of proof of existence theorem. The proof follows the following steps.

(1) Show S-completeness implies $\operatorname{Aut}(x)$ are reductive for all $x \in |\mathfrak{X}|$.
(2) Use local structure theorem to obtain

$$\begin{array}{ccc} [\operatorname{Spec} A/G_x] = (\mathfrak{W}, w) & \xrightarrow{\text{étale}} & (\mathfrak{X}, x) \\ \big\downarrow{\scriptstyle gms} & & \\ \operatorname{Spec} A^{G_x} = \mathbb{W} & & \end{array}$$

Take pullback and obtain

$$\begin{array}{ccc} \mathfrak{W} \times_{\mathfrak{X}} \mathfrak{W} & \xrightarrow[p_1]{} & \mathfrak{W} \\ \big\downarrow{\scriptstyle p_2} & & {\scriptstyle f}\big\downarrow \\ \mathfrak{W} & \xrightarrow{f} & \mathfrak{X} \end{array}$$

where p_1, p_2 are étale. f is affine as $\mathfrak{X}$ has affine diagonal, which means p_1, p_2 are affine.

Consider the following notion of affine GIT, which specializes to the usual GIT when $\mathfrak{W} = BG$ and $\mathbb{W} = *$, where we have

$$\begin{array}{ccc} \operatorname{Spec}_{\mathfrak{W}} \mathcal{A} & \longrightarrow & \mathfrak{W} \\ \big\downarrow{\scriptstyle gms} & & \big\downarrow{\scriptstyle \pi} \quad gms \\ \operatorname{Spec}_{\mathbb{W}} \pi_* \mathcal{A} & \longrightarrow & \mathbb{W} \end{array}$$

This gives us a good moduli space $\operatorname{Spec}\mathcal{B}$ with

$$\begin{array}{ccc} \mathfrak{W} \times_{\mathfrak{X}} \mathfrak{W} & \xrightarrow[p_1,p_2]{} & \mathfrak{W} \\ \Big\downarrow{\scriptstyle gms} & & \Big\downarrow{\scriptstyle gms} \\ \operatorname{Spec}\mathcal{B} & \xrightarrow{q_1,q_2} & \mathbb{W} \end{array}$$

The goal is to show q_1, q_2 étale and that the square is Cartisian. Then we can take $\mathbb{X} = \mathbb{W}/\operatorname{Spec} B$ to be our good moduli space for $\mathfrak{X}$. So we would like to apply Luna fundamental lemma to p_1, p_2 at all points.

(3) S-completeness implies that locally on $\mathbb{W}$, f preserves stabilizer, thus so do p_1 and p_2.
(4) Θ-completeness implies that locally on $\mathbb{W}$,f maps closed points to closed points, but this does not immediately imply p_1, p_2 do the same. One can use Θ-surjectivity to work around this.

□

Theorem 3.30. *The moduli stacks $\overline{\mathcal{M}}_{g,n}(\alpha)$ of α-stable curves admit good moduli space for $\alpha \in (\frac{2}{3} - \varepsilon, 1]$.*

The proof of this theorem follows a similar strategy as the proof of the existence theorem, without using S and Θ-completeness. More direct applications include good moduli spaces for Gieseker semi-stable sheaves, Bridgeland semi-stable objects, K-semi-stable log Fano pairs.

3.4.1. *Intrinsic method vs GIT.* For $\mathcal{M}$ a moduli stack, the GIT approach first embed $\mathcal{M}$ into a quotient $[X/G]$.For example, the moduli of stable curves uses Chow scheme; the moduli of stable sheaves uses Quot scheme. One then study the stability condition with respect to a chosen linearization, so that the interested objects are included in the (semi)stable locus. Then $\mathcal{M} \to X^s/G$ is a coarse moduli space, and $\overline{\mathcal{M}} \to X /\!\!/ G$ is a projective coarse moduli space, and we know exactly how X^s/G compactifies to $X /\!\!/ G$.

The intrinsic approach we looked at uses the following steps.

(1) embed $\mathcal{M}$ into an algebraic stack $\mathfrak{X}$. For example, the moduli of semi-stable objects with respect to some Bridgeland stability condition can be embedded into the moduli stack $\mathcal{M}^{pug}$ of universally gluable perfect complexes.
(2) Show $\mathcal{M} \subseteq \mathfrak{X}$ is open.
(3) Show $\mathcal{M}$ is a quasi-compact algebraic stack (boundedness).
(4) If $\mathcal{M}$ is Deligne-Mumford (so stable is equivalent to semi-stable), prove separatedness using the usual valuative criteria. If not, then prove Θ and S-completeness. Then by existence theorem, we have a coarse moduli space in the first case and good moduli space in the latter.
(5) Verify the existence part of the valuative criteria, which gives properness.
(6) Construct an ample line bundle, making the space projective.

3.5. **Back to non-reductive GIT.**

Definition 3.31. A *topological moduli space* to $\mathfrak{X}$ is $\pi : \mathfrak{X} \to \mathbb{X}$ such that

(1) π is quasi-compact and quasi-separated,
(2) π is universally closed,
(3) for all $x \in |\mathbb{X}|$, there exists a unique closed point $x_0 \in \pi^{-1}(x)$.
(4) For any diagram

$$\begin{array}{ccc} \operatorname{Spec}_{\mathfrak{X}} \mathcal{A} & \xrightarrow{f} & \mathfrak{X} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow \\ \operatorname{Spec}_{\mathbb{X}} \pi_*\mathcal{A} & \longrightarrow & \mathbb{X} \end{array},$$

if f is finite, then p satisfies condition (3).

(5) Condition (4) holds after base change on $\mathbb{X}$.

Topological moduli spaces are categorical, finite, and satisfy Luna fundamental lemma.

Theorem 3.32. *If $\mathfrak{X} \to \mathbb{X}$ is a topological moduli space and closed points have reductive stabilizer, then it is a good moduli space.*

Let $H = U \rtimes R$ be a non-reductive group acting on projective scheme X linearly. Let $\lambda : \mathbb{G}_m \to Z(R)$ act with positive wights on $\operatorname{Lie} U$ (so it is a grading). Then

$$X^{\circ}_{\min}/\hat{U} \to Z_{\min}$$

is a topological moduli space. The assumption $\operatorname{stab}_U(x) = \{e\}$ for $x \in Z_{\min}$ implies all closed points have reductive stabilizer, giving us a good moduli space quotienting $\hat{U}$. To get the quotient for H, we have $[X^{ss,R}_{\min}/H] \to Z^{ss,R}_{\min} /\!\!/ R$ is a topological moduli space, and again can show that it is a good moduli space. Furthermore, taking a character χ for λ, one can prove the $\hat{U}$ theorem in using twisted GIT.