De Rham Cohomologies

Let M be a smooth manifold. The de Rham complex of M is

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots$$

where Ω^i is the space of i-forms.

The de Rham cohomology is

$$H^i_{dR}(M) = H^i_{dR}(M, \mathbb{R}) = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$$

De Rham Theorem

$$H^i_{dR}(M, \mathbb{R}) = H^i_{sing}(M, \mathbb{R})$$

Sheaf theoretic de Rham Theorem

$$H^i_{dR}(M, \mathbb{R}) = H^i(M, \underline{\mathbb{R}}_M)$$

where $\underline{\mathbb{R}}_M$ is the constant sheaf.

Cohomology in the sense of derived functors

Let F be a sheaf of abelian groups on a topological space X.

The sheaf cohomology $H^i(X, \cdot)$ is the i-th derived functor of $\Gamma(X, \cdot)$

Computed as follows

1. Injective resolution

$$0 \to F \to I^0 \to I^1 \to \cdots$$

2. Apply $\Gamma(X, \cdot)$, and throw away $\Gamma(X, F)$

$$0 \to \Gamma(X, I^0) \xrightarrow{d} \Gamma(X, I^1) \xrightarrow{d} \cdots$$

3. Take cohomology

$$H^i(X, F) = R^i\Gamma(X, F) = \frac{\ker d^i}{\ker d^{i-1}}.$$

There are many cohomology theories that compute $H^*(X, \underline{G}_X)$ for G abelian group.

Ex singular cohomology is given by a flasque (instead of injective) resolution

$$0 \to \underline{\mathbb{Q}}_X \to C^0 \to C^1 \to \cdots$$

but one can show $H^i(X, \underline{\mathbb{Q}}_X) = H^i_{sing}(X, \mathbb{Q})$. [Godement]

Ex One way to generalize de Rham cohomology is given by a fine resolution

$$0 \to \underline{\mathbb{R}}_X \to A^0 \to A^1 \to \cdots$$

We have $H^i(X, \underline{\mathbb{R}}_X) = H^i_{dR}(X, \mathbb{R})$ [Gunning, Rossi]

Ex Čech cohomology $\check{H}^i(X, \underline{\mathbb{R}}_X) = H^i(X, \underline{\mathbb{R}}_X)$ [Serre]

Let X be a compact Kähler manifold with Kähler differential Ω.

The Dolbeault cohomology is

$$H^{p,q}(X) = H^q(X, \Lambda^p \Omega)$$

Computed using $A^{p,q}$, differential forms of type p,q. locally given by

$$\sum f_{i_1,\dots,i_p,j_1,\dots,j_q}\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

This gives a fine resolution

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to \cdots$$

and its cohomology is $H^{p,q}(X)$.

Hodge Theorem

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \quad \text{with symmetry } H^{p,q}(X) = H^{q,p}(X)$$

Hypercohomology.

Let $F^\bullet$ be a complex of sheaves on X

$$F^\bullet = [F^0 \xrightarrow{d} F^1 \xrightarrow{d} F^2 \xrightarrow{d} F^3 \to \cdots], \quad d^2 = 0.$$

The (naive) cohomology of $F^\bullet$ is $H^i(F^\bullet) = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$

A map $\varphi: F^\bullet \to G^\bullet$ is a quasi-isomorphism if $H^i(F^\bullet) \overset{H^i(\varphi)}{\cong} H^i(G^\bullet)$

We can also view $F^\bullet$ as an object in the derived category

Then we can take the derived functor cohomology of $F^\bullet$ as before:

1. Injective resolution

$$F^\bullet \xrightarrow{\varphi} I^\bullet \quad \text{s.t. } I^p \text{ are injective, } \varphi \text{ is quasi-isomorphism}$$

2. Apply $\Gamma(X, \cdot)$ and get

$$\Gamma(X, I^\bullet) \quad \text{a complex of abelian groups.}$$

3. Take cohomology.

$$\mathbb{H}(X, F^\bullet) := H^i(\Gamma(X, I^\bullet))$$

This is called the hypercohomology. It can be computed by spectral sequence

$$E_1^{pq} = H^q(X, F^p) \Rightarrow \mathbb{H}^*(X, F^\bullet)$$

Let $f: X \to Y$ be a map of topological spaces

Then $\mathbb{H}^*(Y, Rf_* F) = H^*(X, F)$ for any sheaf F.

so we have spectral sequence $H^p(Y, R^q f_* F) \Rightarrow H^{p+q}(X, F)$.

Suppose $U \overset{f}{\hookrightarrow} X$ with $X - U = D_1 \cup \cdots \cup D_k$ simple normal crossing.

Then $H^*(U, \underline{\mathbb{C}}_U) = \mathbb{H}^*(X, Rf_* \underline{\mathbb{C}}_U)$

$\underline{\mathbb{C}}_U$ has resolution $\mathbb{C}_U \to \Omega^0_U \to \Omega^1_U \to \cdots$

so we need

$$H^*(U, \mathbb{C}_U) = \mathbb{H}^*(U, [\Omega^0_U \to \Omega^1_U \to \cdots])$$
$$= \mathbb{H}^*(X, [f_* \Omega^0_U \to f_* \Omega^1_U \to \cdots])$$

Define logarithmic poles $\Omega^p(\log D)$ to be sheaf on X with stalks

$$(\Omega^1_X(\log D))_x = \mathcal{O}_{X,x} \frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{X,x} \frac{dz_k}{z_k} \oplus \mathcal{O}_{X,x} dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{X,x} dz_n.$$

$\Omega^p_X(\log D) = \Lambda^p \Omega^1(\log D)$ where D is locally given by $z_1 \cdots z_k = 0$ near x.

Thm $\exists$ quasi-isomorphism
$$\Omega_X^\bullet(\log D) \to f_* \Omega_U^\bullet.$$

Hartshorne's Algebraic de Rham cohomology

Completions Let X be a noetherian scheme or complex analytic space.
$Y \subseteq X$ a subscheme with ideal sheaf I.

The formal completion $\hat{X}$ of X along Y.
is the locally ringed space
$$\hat{X} = (Y, \varprojlim \mathcal{O}_X / I^r).$$

Ex The formal completion of $\mathbb{C}^n$ along 0 is
$$\operatorname{Spec} \mathbb{C}[[x_1, x_2, \dots, x_n]]$$
it is topologically just the point $\{0\}$, but the sheaf keeps track of all Taylor series at 0.

In general, think of formal completion as enriching Y with all infinitesimal data of X. similar to the normal bundle gives Y the linear data of X.

We have a closed embedding $i : \hat{X} \to X$.
If F is a coherent sheaf on X, then $\hat{F} := i^* F$ is the induced sheaf on $\hat{X}$
We have $i^* F \cong \varprojlim (F / I^r F)$
so think of $\hat{F}$ as "restriction of F to $\hat{X}$".

Let Y be a finite type $\mathbb{C}$-scheme, $\mathbb{H}^*(Y, \Omega_Y^\bullet)$ does not retrieve $H^*_{sing}(Y)$ in general.
The idea is that $\Omega_Y^\bullet$ does not contain enough infinitesimal information about Y.

Suppose $Y \hookrightarrow X$ is a closed embedding with X smooth, then we define
$$H^*_{dR}(Y) := \mathbb{H}^*(\hat{X}, \hat{\Omega}_X^\bullet)$$
to be the algebraic de Rham cohomology

This definition is independent of $Y \hookrightarrow X$. and is natural in the sense that it is a contravariant functor, induces long exact sequences, etc.

<u>Comparison theorem</u> [Hartshorne]

$\exists$ natural isomorphisms

$$H^i_{dR}(Y) \cong H^i_{sing}(Y, \mathbb{C})$$

Chiral de Rham cohomology

In the study of conformal field theory,

Malikov-Schechtman-Vaintrob defined the Chiral de Rham complex as a sheaf of vertex algebras $\Omega^{ch,\bullet}_X$ which is quasi-isomorphic to $\Omega^\bullet_X$ for X smooth.

For a $\mathcal{D}$-module M, one associates a deRham complex

$$DR(M) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^\bullet, M).$$

Let LX be the formal loop space of X, containing the arc space $L^+X \xhookrightarrow{i} LX$ then the chiral de Rham cohomology is

$$H^*_{cdR}(X) = \mathbb{H}^*(DR(i_\bullet \omega_{L^+X}))$$

for X affine with étale map $X \to \mathbb{A}^d$.

where $i_\bullet$ is the $\mathcal{D}$-module pushforward.

Derived de Rham cohomology

Let B be a $\mathbb{C}$-algebra with standard simplicial resolution $P_\bullet \to B$.

Apply $\Omega^*_\bullet$ to $P^\bullet$ we get a double complex $\Omega^*_{P_\bullet}$

The total complex is $L\Omega^*_B$ the cotangent complex.

This is q.i to the trivial complex, so it is not useful.

To define the cohomology correctly, we need the Hodge filtration.

$$F^i \Omega^*_{P_\bullet} = \bigoplus_{q \geq i} \Omega^q_{P_\bullet}$$

Then the de Rham complex is taken to be

$$L\hat{\Omega}^*_B := \varprojlim L\Omega^*_B / F^N$$
$$= \varprojlim \mathrm{Tot}(\Omega^{<N}_{P_\bullet}).$$