Equivariant Segre-Verlinde on Quot schemes.

① Universal series on $\mathrm{Quot}_S(E, n)$ for $E = \bigoplus_{i=1}^{N} \mathcal{O}_S\langle y_i\rangle$, $V = \bigoplus_{i=1}^{r} \mathcal{O}_S\langle v_i\rangle$

Compact case

$$S_S(E, V; q) = A_1(q)^{c_1(S)c_1(V)} \cdot A_2(q)^{c_1(S)^2} \cdot A_3(q)^{c_1(S)\cdot c_1(E)}$$

Non compact case

$$S_{\mathbb{C}^2}(E, V; q) = \prod_{\substack{\mu, \nu, \xi \\ \text{partitions}}} A_{\mu,\nu,\xi}(q)^{\int_S c_1(S)\cdot c_\mu(V)\cdot c_\nu(S)\cdot c_\xi(E)}$$

Degree of $\int_S c_1(S)\cdot c_\mu(\alpha)\cdot c_\nu(S)\cdot c_\xi(E)$ in λ_1, λ_2

is $|\mu| + |\nu| + |\xi| - 1$

so degree 0 occurs if one of

μ, ν, ξ is (1), rest are (0)

$$A_{(1),0,0}(q)^{\int_S c_1(S)c_1(\alpha)}, \quad A_{0,(1),0}(q)^{\int c_1(S)^2}$$

$$A_{0,0,(1)}(q)^{\int c_1(S)c_1(E)}.$$

Want to compare

$$A_{100},\ A_{010},\ A_{001}$$

to $\quad A_1,\ A_2,\ A_3$

② Relate $A_{\mu,\nu,\xi}$ to A_1, A_2, A_3

<u>Ex</u> Integrating over compact toric S with fixed point

$p_1, \dots, p_M$, with weights of T_{p_i}

given by $a_1^{(i)}, a_2^{(i)}$

Then

$$\int_S \gamma = \sum_{i=1}^{M} \frac{\gamma|_{p_i}}{e(T_{p_i})} = \sum \frac{\gamma|_{p_i}}{a_1^{(i)} a_2^{(i)}}$$

For $V = \bigoplus_1^r \mathcal{O}_S\langle v_i\rangle$, $E = \bigoplus^N \mathcal{O}_S\langle y_i\rangle$

$$\int_S c_\mu(V) c_\nu(S) c_\xi(E) = \sum_{i=1}^{M} \left.\frac{e_\nu(\lambda_1, \lambda_2)}{\lambda_1 \lambda_2}\right|_{\lambda = a^{(i)}} \cdot c_\mu(V) c_\xi(E)$$

Then $\sum q^n \int_{[\mathrm{Quot}_S]^{\mathrm{vir}}} s(V^{[n]})$

$$= \left[\prod_{i=1}^{M} \sum q^n \int_{[\mathrm{Quot}_{\mathbb{C}^2}]^{\mathrm{vir}}} s(V^{[n]}) \right]_{\lambda = a^{(i)}}$$

$$= \prod_{i=1}^{M} \prod_{\mu,\nu,\xi} A_{\mu,\nu,\xi}(q)^{\int_{\mathbb{C}^2} c_1(S) c_\mu(V) c_\nu(S) c_\xi(\Sigma)} \Big|_{\lambda = a^{(i)}}$$

$$= \prod_{\mu,\nu,\xi} A_{\mu,\nu,\xi}(q)^{\int_S c_1(S) c_\mu(V) c_\nu(S) c_\xi(\Sigma)}$$

The compact contribution are from the degree 0 part which is exactly when $|\mu| + |\nu| + |\xi| - 1 = 0$.

$$\Rightarrow A_1 = A_{(1),0,0}, \quad A_2 = A_{0,(1),0}, \quad A_3 = A_{0,0,(1)}.$$

③ Weak Segre-Verlinde correspondence

By Compact S-V, which says

$$A_i(q) = B_i((-1)^N q)$$

we have $A_{\mu\nu\xi}(q) = B_{\mu\nu\xi}((-1)^N q)$

for deg 0 part.

Over $\mathbb{C}^2$,

$$S = \prod_{\deg -1,1,2,\dots} A_{\mu,\nu,\xi}^{\int_S c_1 \cdots} \cdot \prod_{\deg 0} A_{\mu,\nu,\xi}^{\int_S c_1 \cdots}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$V = \prod_{\deg -1,1,2,\dots} B_{\mu,\nu,\xi}^{\int_S c_1 \cdots} \cdot \prod_{0} B_{\mu\nu,\xi}^{\int c_1 \cdots}$$

deg 0 part of S-V might come from product of deg -1 with deg +1 giving a multiple of $\left(\int_S c_1(S)\right)^2$.

so $(S-V)|_{\deg 0} = \sum q^n \left(\int_S c_1(S)\right)^2 \cdot \frac{f_n}{\lambda_1 \lambda_2^{(n-2)}}$

for some $f_n \in H^*_T(pt)$

④ Proof for univ. series expansion $S=\mathbb{C}^2$

Let $H^*_{T_1}(pt) = \mathbb{C}[w_1, \ldots, w_r]$

$H^*_{T_2}(pt) = \mathbb{C}[m_1, \ldots, m_N]$

Chern roots of $V^{[n]}$ for $V = \bigoplus_{i=1}^r \mathcal{O}_S \langle v_i \rangle$

over $\mu = (\mu^{(1)}, \ldots, \mu^{(N)})$, $\dim V^{[n]} = nr$, are

$$\bigcup_{j=1}^{r} \bigcup_{i=1}^{N} \bigcup_{\square \in \mu^{(i)}} \{ w_j + m_i - c(\square)\lambda_1 - r(\square)\lambda_2 \}$$

so $\sum q^n \int_{[\mathrm{Quot}]^{vir}} S(V^{[n]})$

$$= \sum_{|\mu|=n} q^n \frac{\prod_{i=1}^{N} \prod_{\square \in \mu^{(i)}} \prod_{j=1}^{r} (1 + w_j + m_i - c(\square)\lambda_1 - r(\square)\lambda_2)}{e(T^{vir}_{Z_\mu})}$$

Similar expression for Verlinde.

Let $I_S^N(V;q,z)$

$$= \sum_{n=0}^{\infty} \left((-1)^N q^n\right) \chi^{vir}\left(\mathrm{Quot},\ \Lambda_{-z} V^{[n]} \otimes \det(\mathcal{O}_S^{[n]})^{-1}\right)$$

Then $\log I_S^N(V;q,z)$ expand

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} H_{j,k}(q,z,\vec{w},\vec{m})\, \lambda_1^j \lambda_2^k$$

Then $H_{j,k} \in \mathbb{Q}((m_1,\dots,m_N))[[q,z,w_1,\dots,w_r]]$.

Expand in elementary symmetric polynomial basis for $\vec{w}$.

$$= \sum_{\substack{j,k \\ \mu \text{ partition}}} H_{\mu,j,k}(q,z,\vec{m})\, \lambda_1^j \lambda_2^k \ c_\mu(V)$$

Use this expression to

compute for compact $S = \mathbb{P}^1 \times \mathbb{P}^1$.

fixed-points are $p_{\alpha\beta} : \alpha,\beta \in \{1,0\}$

$$a_1^{(\alpha\beta)} = (-1)^\alpha \lambda_1,\quad a_2^{(\alpha\beta)} = (-1)^\beta \lambda_2.$$

If $H_{\mu,-2,-2} \neq 0$

then result for S' has term

$$\sum_{ab} H_{\mu,-2,-2} \lambda_1^{-2} \lambda_2^{-2} \Big|_{\lambda_1 = \lambda_1^{(ab)}, \lambda_2 = \lambda_2^{(ab)}})$$

$$= 4 \cdot H_{\mu,-2,-2} \cdot \lambda_1^{-2} \cdot \lambda_2^{-2}$$

Since equivariant pushforward for compact surface

lands in $H_T^*(pt) = \mathbb{C}[\lambda_1, \lambda_2]$

this is not possible

This works for any $H_{\mu,j,k}$ where j,k even

But can have linear dependence

$$\sum_i \lambda_1^j \lambda_2^k \Big|_{\lambda = a^{(i)}} = \sum_i \lambda_1^{j'} \lambda_2^{k'} \Big|_{\lambda = a^{(i)}}$$

for j, k, j', k'

or worse, sometimes $\sum_i \lambda_1^j \lambda_2^k \Big|_{\lambda=a^{(i)}} = 0$

so can't conclude $H=0$ yet.

For this, replace w_i by $w_i \cdot (\lambda_1+\lambda_2)^l$

then $\sum_i H_{\mu,j,k}(q,z;\vec{m})\lambda_1^j\lambda_2^k(\lambda_1+\lambda_2)^l \Big|_{\lambda=a^{(i)}}$

must be polynomial in λ_1, λ_2.

Get extra restrictions and conclude

$H_{\mu,j,k}=0$ for any $\min(j,k)\le -2$

when $r>1$

Similarly show $H(q,z,\vec{m})$ power series in m

$$\Rightarrow \log I_S^N(V;q,z)$$

$$= \sum_{\substack{j,k\geq 0\\ \nu,\xi}} H_{\nu,\xi,j,k}(q,z) \frac{\lambda_1^j \lambda_2^k c_\nu(S) c_\xi(E)}{\lambda_1 \lambda_2}$$

Use symmetry in λ_1, λ_2

$$= \sum_{\mu,\nu,\xi} H_{\mu,\nu,\xi}(q,z) \int_S c_\mu(V) c_\nu(S) c_\xi(E)$$

Note in Hilb case, Obstruction is $K_S^{[n]}$

so $e(\mathrm{Obs})$ has factor of $c_1(S)$

similar for Quot.

$$\Rightarrow \sum H_{\mu,\nu,\xi} \int_S c_1(S) c_\mu c_\nu c_\xi.$$

Take exp and get desired expression.

Why use I_5^N ?

because need to generalize to arbitrary rank which might not work for Verlinde numbers.

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⑤ Conjecture for CY4 version has the multiple of $c_1(S)$ replaced by $c_3(X)$.

For reduced invariants on K-trivial surface,

$S=\mathbb{C}^2$, $T=\{(t_1,t_2): t_1t_2=1\}$, $N=1$,

$\text{red}.\, S_0 = A_2(q)\, c_2 + A_1(q)\, c_1^2 + A_0(q)$

$\text{red}.\, V_0 = B_1(q) c_1^2 + A_0(q)$

Then in deg 0, rank $-k$ Segre

rank $r=k-1$ Verlinde

<u>Thm</u> $k^2 A_1(q) + \binom{|k|}{2} A_2(q) = rk\, B_1(q)$