Some background

Let k be an algebraically closed field.

Defn: *Affine n-space over k*, denoted $\mathbb{A}^n$, is the set of n-tuples of elements of k. An element of $\mathbb{A}^n$ is called a *point*.

Defn: *Projective n-space over k*, denoted $\mathbb{P}^n$, is defined as the set of equivalence classes of $n + 1$-tuples $(a_0, ... a_n)$ with entries from k, under the equivalence relation $(a_0, ... a_n) \sim (\lambda a_0, ... \lambda a_n)$, for $\lambda \neq 0$. Essentially, $\mathbb{P}^n$ takes each line passing through the origin of $\mathbb{A}^n$ as one point.

Defn: A set $Y \subset \mathbb{A}^n$ (resp. $Y \subset \mathbb{P}^n$) is *algebraic* when it is the zero set of some set of polynomials (resp. homogenous polynomials) T. (We'll denote the zero set of T by $Z(T)$).

Defn: The *Zariski topology* is defined by taking the open sets to be the complements of the algebraic sets. Note that it is indeed a toplopgy, i.e. the intersection of two open sets is open (since the union of two algebraic sets is algebraic) and the union of any family of open sets is open (since the intersection of any family of algebraic sets is algebraic).

Defn: A subset Y of a topological space is *irreducible* when it cannot be expressed as the union of two proper subsets both closed in Y.

Defn: An affine or projective *variety* is an irreducible set under the Zariski topology. Varieties are basically the zero sets of irreducible polynomials.

In [1]:
```
reset()
%display latex
Partitions.options.latex="list"
load("blowup.sage")
# p = y^2-x^2*(x+1)
p = y^2-x^7
c = Curve(p)
show(p)
```

Out[1]: $-x^7 + y^2$

Singularities

For a plane curve given by the equation $f(x, y) = 0$, the regular points are the points where the curve is smooth. This is equivalent to the points on the curve where the partial derivatives of f do not both vanish. The singular points (a.k.a. singularities) are the points on the curve that aren't regular, namely where

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad f(x, y) = 0.$$

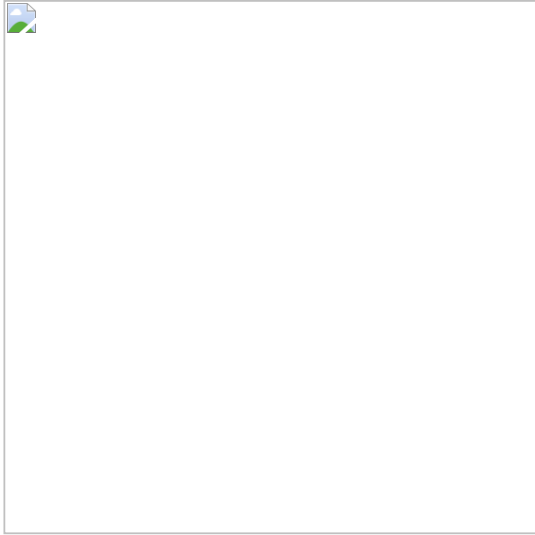The code below computes and prints these points for the curve given above:

In [2]:
```
show(latex(c.getSingularities()))
```

Out[2]: $[[x = 0, y = 0]]$

Resolution of Singularities

We can "resolve" singularities (i.e. make them smooth) via blow-ups.

Defn: The *blow-up* of $\mathbb{A}^n$ at 0 is the (closed) subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = y_i x_j | i, j = 1, ... n\}$.

There is a natural morphism $\phi : X \to \mathbb{A}^n$ given by the projection to the affine part of an element of X.

A few notes:

1. Let P be a point in $\mathbb{A}^n$. If $P \neq O$, then $\phi^{-1}(P)$ is a single point, since we can uniquely determine the homogenous coordinates for the projective part. Indeed, if $a_i \neq 0$, then for each j, $y_j = \frac{a_j}{a_i} y_i$. We can set y_i to be anything other than 0, and this yields the same homogenous coordinate (any such choice of y_i is just a scalar multiple of any other choice). In particular, setting $y_i = a_i$ gives $(y_1, \dots y_n) = (a_1, \dots a_n)$.

2. $\phi^{-1}(O) \cong \mathbb{P}^{n-1}$ since the homogenous coordinates of the projective part are subject to no restriction.

3. The blow-up of $\mathbb{A}^n$ looks like $\mathbb{A}^n$, except the point O has been replaced by a copy of $\mathbb{P}^{n-1}$, denoted as the exceptional divisor.

4. Let L be a line passing through O, given by parametric equations $x_i = a_i t$ with a set of a_i not all 0 and where $t \in \mathbb{A}^1$. Let $L' = \phi^{-1}(L - O) \in X - \phi^{-1}(O)$. L' can be written parametrically as $x_i = a_i t$, $y_i = a_i$ (which is true because the y_i are homogenous coordinates). For $t = 0$, this parameterization still makes sense, so altogether, it gives the closure $\bar{L}'$ of L' in X. Note that the slope of the line determines $(a_1, \dots a_n)$ (i.e. $\bar{L}$ intersects $\phi^{-1}(O)$ according to its slope).

Defn: Let Y be a closed subvariety of $\mathbb{A}^n$ passing through O (we can just think of Y as a polynomial curve passing through O). The *blow-up of Y at the point O* is given by $\tilde{Y} = \phi^{-1}(\bar{Y} - O)$ (closure of the inverse image of Y, analogous to above?????), with ϕ as defined above. $\tilde{Y}$ is called the *strict transform* (moving forward, we'll denote all strict transforms with tildes), and when we compute the total inverse image of Y under ϕ, we also get a copy of E, which intersects $\tilde{Y}$ according to the slope(s) of Y where it passes through O. We explore this in more depth in the example below.

Blow-ups of $\mathbb{C}^2$

Let $[u : v]$ be the homogeneous coordinates in the blow-up of $\mathbb{C}^2$ at O, which we will denote $X \subseteq \mathbb{C}^2 \times \mathbb{P}^1$. The blow-up of $\mathbb{C}^2$ is covered by two sets, namely $U = \{u \neq 0\}$ and $V = \{v \neq 0\}$. With respect to this cover, there are two charts:

$\phi_1 : U \to Bl_O \mathbb{C}^2$ given by $(x, y)[u : v] \mapsto (x, \frac{v}{u})$ and $\phi_2 : U \to Bl_O \mathbb{C}^2$ given by $(x, y)[u : v] \mapsto (y, \frac{u}{v})$.

Note that in ϕ_1, the choice of coordinates comes from the fact that we can write y in terms of x and $\frac{v}{u}$ using the equation $xv = yu$ (from the definition of the blow-up). A similar argument applies to the choice of coordinates for ϕ_2. Also note that [insert note about which points are "captured" by which chart]

To blow up a curve C, consider the map $\psi : X \to \mathbb{C}^2$ given by $(x, y)[u : v] \mapsto (x, y)$ (same as ϕ defined above; we changed the name to avoid confusion). We want to compute $\phi_1(\psi^{-1}(C) \cap U)$ and $\phi_2(\psi^{-1}(C) \cap V)$. Note that $\psi^{-1}(C)$ imbeds C into $\mathbb{C}^2 \times \mathbb{P}^1$ without restricting the homogeneous part, and applying ϕ_1 and ϕ_2 restricts the imbedding to X since the two charts come from the equation $xv = yu$. $\mathbb{P}^1$ can be thought of as a circle, and each chart covers all but one point, so the idea is that for each chart, we can remove a point from the circle, flatten it into a line, and treat the homogeonous coordinates as an affine coordinate (hence why the charts map to $\mathbb{C}^2$). Together, the images of C under these two charts make up $\tilde{C}$.

Once we blow up C, if singularities still exist in the image of C under either chart, we continue the process.

Computing the charts

The code below computes the equations of the images of C under the two charts at each singularity. Note that in the equations, we perform the change of variables $(x, \frac{v}{u}) \mapsto (x_1, y_1)$ and $(y, \frac{u}{v}) \mapsto (x_2, y_2)$. This allows us to perform the substitutions $x = x_1, y = x_1 y_1$ (which comes from the fact that $y = x\frac{v}{u}$) into the original polynomial to get the polynomial under the first chart with the change of variables. Likewise, the polynomial under the second chart is given by substituting $y = x_2, x = x_2 y_2$ into the original polynomial.

`In [3]:`
```
chartpolys = c.computeCharts(x1, y1, x2, y2)
chartpolys
img = chartpolys[0][0]
```

`Out[3]:` Original polynomial:

$$-x^7 + y^2$$

Singularity:

$$[x = 0, y = 0]$$

Polynomial under first chart (first homogenous coordinate is nonzero):

$$-x_1^7 + x_1^2 y_1^2$$

Polynomial under second chart (second homogenous coordinate is nonzero):

$$-x_2^7 y_2^7 + x_2^2$$

Computing the strict transform and exceptional divisor

Next, we can factor the images of C under each chart to get $\tilde{C}$ and E. E will be a factor of x_n raised to some power, and $\tilde{C}$ is everything else. In the example below, we're looking at the image of C under ϕ_1. We can also compute the number N_E, which is number of copies of the exceptional divisor we have, and corresponds to the power the factor of x_n is raised to.

`In [4]:`
```
stne1 = computeStrictTransformAndExceptionalDivisor(img, x1, 0)
```

`Out[4]:` Original polynomial:
$$-x_1^7 + x_1^2 y_1^2$$

$$E : \{x_1 = 0\}$$

$$\tilde{C} : \{-x_1^5 + y_1^2 = 0\}$$

$$N_E = 2$$

Resolving further singularities

In this example, $\tilde{C}$ under ϕ_1 still contains a singularity, so we need to blow it up again. Note that when looking for singularities in $\tilde{C}$, we only need to check points that intersect with E (i.e. points where $x_n = 0$), since these are the points at the original singularity point. We already know that points away from the singularity were regular to begin with, so even if singularities appear at those points in blow-ups, we can ignore them.

`In [5]:`
```
c1 = Curve(stne1[0], x1, y1)
show(c1.polynomial)
c1.getSingularitiesWithConstraint()
show(c1.singularities)
```

`Out[5]:` $-x_1^5 + y_1^2$

$$[[x_1 = 0, y_1 = 0]]$$

In [6]:
```
chartpolys = c1.computeCharts(x_3, y_3, x_4, y_4)
```

Out[6]: Original polynomial:

$$-x_1^5 + y_1^2$$

Singularity:

$$[x_1 = 0, y_1 = 0]$$

Polynomial under first chart (first homogenous coordinate is nonzero):

$$-x_3^5 + x_3^2 y_3^2$$

Polynomial under second chart (second homogenous coordinate is nonzero):

$$-x_4^5 y_4^5 + x_4^2$$

In [7]:
```
stne2 = computeStrictTransformAndExceptionalDivisor(chartpolys[0][0], x3, stne1[1])
```

Out[7]: Original polynomial:

$$-x_3^5 + x_3^2 y_3^2$$

$$E : \{x_3 = 0\}$$

$$\tilde{C} : \{-x_3^3 + y_3^2 = 0\}$$

$$N_E = 4$$

The function below takes a polynomial curve and goes through all the steps until its singularities are fully resolved.

In [8]:
```
fullyResolveSingularities(p)
```

Out[8]: Original polynomial:

$$-x^7 + y^2$$

Singularity:

$$[x = 0, y = 0]$$

Polynomial under first chart (first homogenous coordinate is nonzero):

$$-x_1^7 + x_1^2 y_1^2$$

Polynomial under second chart (second homogenous coordinate is nonzero):

$$-x_2^7 y_2^7 + x_2^2$$

Original polynomial:

$$-x_1^7 + x_1^2 y_1^2$$

$$E : \{x_1 = 0\}$$

$$\tilde{C} : \{-x_1^5 + y_1^2 = 0\}$$

$$N_E = 2$$

Original polynomial:

$$-x_1^5 + y_1^2$$

Singularity:

$[x_1 = 0, y_1 = 0]$

Polynomial under first chart (first homogenous coordinate is nonzero):

$-x_3^5 + x_3^2 y_3^2$

Polynomial under second chart (second homogenous coordinate is nonzero):

$-x_4^5 y_4^5 + x_4^2$

Original polynomial:

$-x_3^5 + x_3^2 y_3^2$

$E : \{x_3 = 0\}$

$\tilde{C} : \{-x_3^3 + y_3^2 = 0\}$

$N_E = 4$

Original polynomial:

$-x_3^3 + y_3^2$

Singularity:

$[x_3 = 0, y_3 = 0]$

Polynomial under first chart (first homogenous coordinate is nonzero):

$x_7^2 y_7^2 - x_7^3$

Polynomial under second chart (second homogenous coordinate is nonzero):

$-x_8^3 y_8^3 + x_8^2$

Original polynomial:

$x_7^2 y_7^2 - x_7^3$

$E : \{x_7 = 0\}$

$\tilde{C} : \{y_7^2 - x_7 = 0\}$

$N_E = 6$

No singularities for $\tilde{C} = y_7^2 - x_7$ that intersect the exceptional divisor

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Original polynomial:

$-x_8^3 y_8^3 + x_8^2$

$E : \{x_8 = 0\}$

$\tilde{C} : \{-x_8 y_8^3 + 1 = 0\}$

$N_E = 6$

No singularities for $\tilde{C} = -x_8 y_8^3 + 1$ that intersect the exceptional divisor

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Original polynomial:
```
$-x_4^5 y_4^5 + x_4^2$

$E : \{x_4 = 0\}$

$\tilde{C} : \{-x_4^3 y_4^5 + 1 = 0\}$

$N_E = 4$

```
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```
No singularities for $\tilde{C} = -x_4^3 y_4^5 + 1$ that intersect the exceptional divisor

```
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Original polynomial:
```
$-x_2^7 y_2^7 + x_2^2$

$E : \{x_2 = 0\}$

$\tilde{C} : \{-x_2^5 y_2^7 + 1 = 0\}$

$N_E = 2$

```
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```
No singularities for $\tilde{C} = -x_1^5 + y_1^2$ that intersect the exceptional divisor

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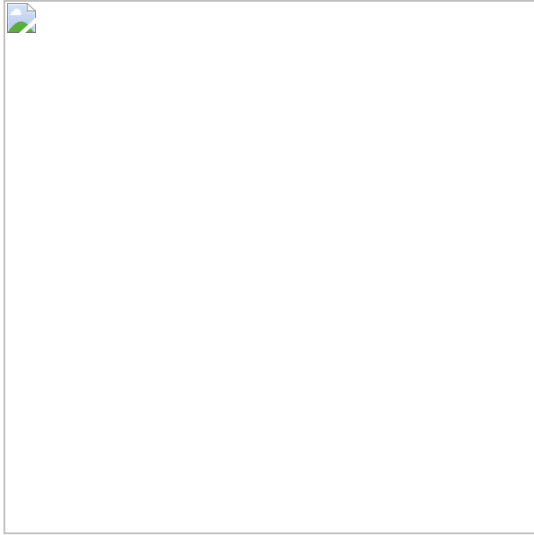
Labelling Exceptional Divisors

In the examples above, the labelling of the coordinates is a bit arbitrary. If we think of the coordinates used in the process of resolving singularities in a binary tree (where each copy of $(C)^2$ is a node, and a node's children are itself under each chart, with the left child corresponding to the chart where the first homogenous coordinate is nonzero, and the right child corresponding to the chart where the second homogenous coordinate is nonzero), then for each copy of $(C)^2$, n in the coordinate system is the index of the corresponding node if the binary tree were stored in an array. This labelling is a bit confusing, and mostly exists to make writing the code easier. (It also breaks down when we have multiple singularities.)

Fortunately, there are better ways to label exceptional divisors, and the coordinates associated with them. Consider the figure below:
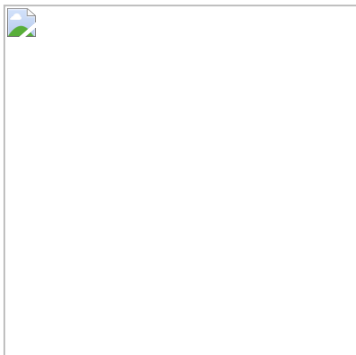

Here, C is the curve and L is a line not tangent to C. If we blow it up, we get something that looks like this:

We label C as $E_{0,1}$ and L as $E_{1,0}$. We get another exceptional divisor in this blow-up, which we label $E_{1,1}$ by adding the labellings of C and L.

Defn: Let L and L' be two divisors (curves) intersecting at a point P. The *divisors between L and L'* are the exceptional divisors that result from blowing up P (and inductively, intersection points of L, L', and previous exceptional divisors).

Setting $E_{0,1}$ and $E_{1,0}$ as above, we can inductively label the exceptional divisors that appear in our blow-ups, where $E_{\kappa+\kappa',r+r'}$ is the exceptional divisor that appears in the blow-up of the intersection of $E_{\kappa,r}$ and $E_{\kappa',r'}$. In this way, as we blow up intersections of exceptional divisors, we construct a *Stern-Brocot tree*.
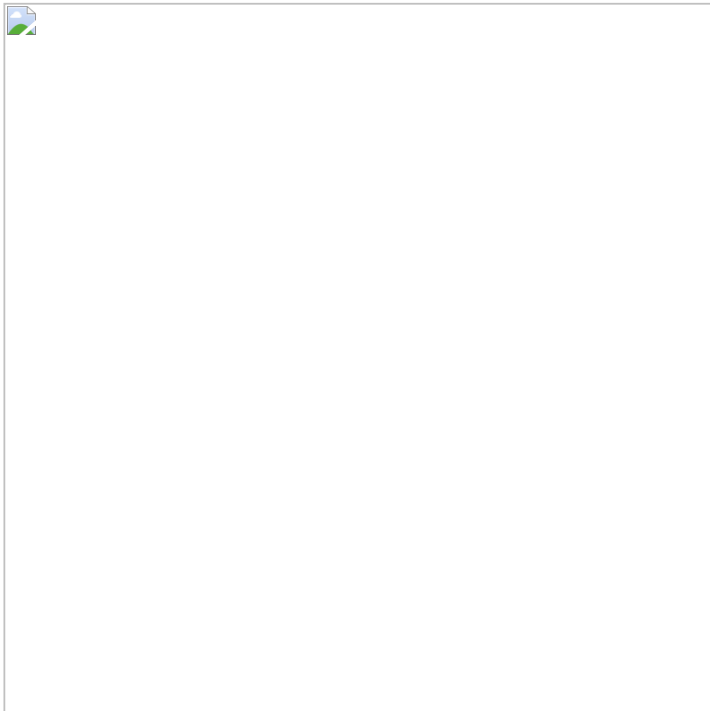
In this tree, two vertices are connected if they intersect in the graph of a blow-up. The fractions (orange) are constructed by $\frac{\kappa}{\kappa+r}$ for $E_{\kappa,r}$. Notice that they increase from left to right, and are always in lowest terms (i.e. the numerator and denominator are coprime).

The values of κ and r also give us information about the coordinate system. If we denote the starting coordinate system (x, y), then the change of coordinates to where $E_{\kappa,r}$ first appears is given by $(x, y) = (\tilde{x}^r \tilde{y}^a, \tilde{x}^\kappa \tilde{y}^b)$, where a, b are integers that can be uniquely determined given κ, r.

In [ ]: