Based on joint work with Eduardo de Lorenzo Poza

Arc-Floer conjecture.

For a hypersurface singularity $\{f=0\} \subseteq \mathbb{C}^n$

there are two objects associated to it often used in singularity theory.

Contact loci	and Milnor fiber
- defined using arc/jet space	- can be made into Liouville domain
- studied via motivic integration	- has monodromy action

mth jet space is $\mathcal{L}_m(\mathbb{C}^n) = \{\gamma : \operatorname{Spec} \mathbb{C}[t]/t^{m+1} \to \mathbb{C}^n\}$

m-th restricted contact locus of f is

$$\mathcal{X}_m = \{\gamma : \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}^n \mid \gamma(0)=0,\ f(\gamma(t)) = t^m \bmod t^{m+1}\}$$

Milnor fiberation is

$$\frac{f}{|f|} : S_\varepsilon - f^{-1}(0) \longrightarrow S^1 \qquad \text{for } 0<\varepsilon \ll 1$$

gives monodromy φ on the fiber.

Connection between the two obj:

<u>Thm</u> (Denef, Loeser 2002)

For $m \geq 1$,

$$\underset{\text{lefschetz num}}{\Lambda(\varphi^m)} = \underset{\text{euler char.}}{\chi(\mathcal{X}_m)}$$

<u>Rmk</u> For a symplectomorphism ϕ, the Floer homology $HF_*(\phi)$ satisfies

$$\Lambda(\phi) = \chi_{HF}(\phi)$$

so one expects relation between

$$H_c^*(\mathcal{X}_m) \quad \text{and} \quad HF_*(\varphi^m)$$ fixed pt Floer homology.

Since Milnor fiber is Liouville domain,

we use $HF_*(\varphi^m, +)$, $+$ indicates a slope near boundary

<u>Conjecture</u> Let $f: \mathbb{C}^n \to \mathbb{C}$ have an isolated singularity at 0

then for $m \geq 1$,

$$HF_*(\varphi^m, +) \cong H_c^{*+(n-1)(2m+1)}(\mathcal{X}_m, \mathbb{Z})$$

<u>Thm</u> (de la Bodega, de Lorenzo Poza)

conjecture holds when $n=2$.

(Budur, de Bobadilla)

holds when $m = \mathrm{mult}(f)$.

(de Lorenzo Poza, H.)

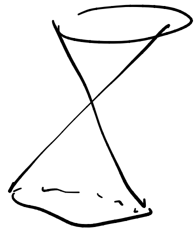
holds when f is homogeneous.

Milnor fiber

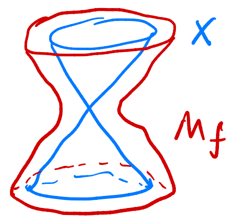
Let $f \in \mathbb{C}[x_1, \dots, x_n]$ homogeneous. isolated singularity

$S = \{f=0\} \subseteq \mathbb{P}^{n-1}$ is a smooth hypersurface

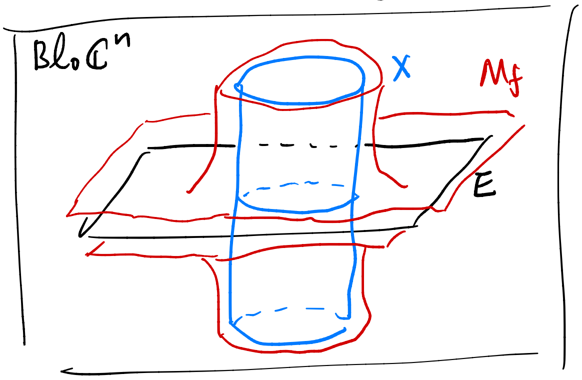
$X = \{f=0\} \subseteq \mathbb{C}^n$ is affine cone of S.

$X \simeq$ 

Prop Milnor fiber of X is diffeo to $f^{-1}(1)$



Resolve X by Blowing up origin: $\mu : Bl_0\mathbb{C}^n \to \mathbb{C}^n$

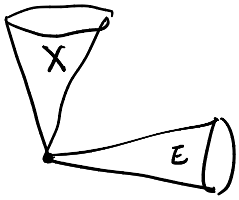


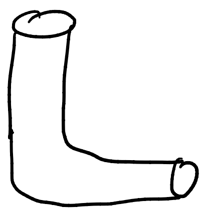
$M_f = \mu^{-1}(1)$

$X \cup E = \mu^{-1}(0)$

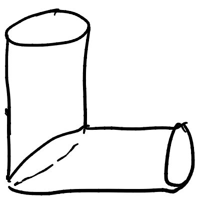
M_f is described using A'Campo model. F

In the curve case, consider two copies of $\mathbb{C}$ intersecting at origin

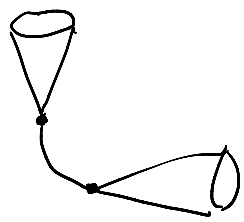


and M_f should look like 

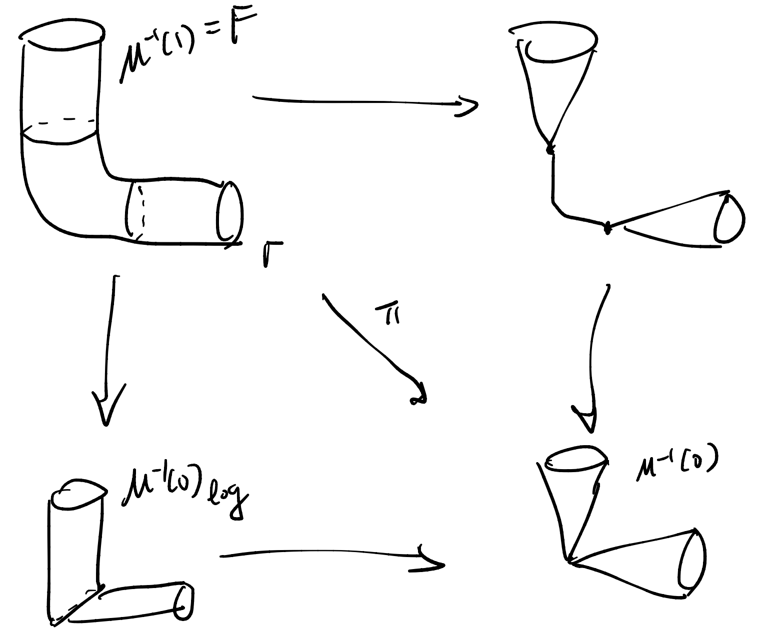
Consider Kato-Nakayama space $(Bl_0\mathbb{C}^2, X \cup E)_{log}$.



but this is not very smooth. so take



and get

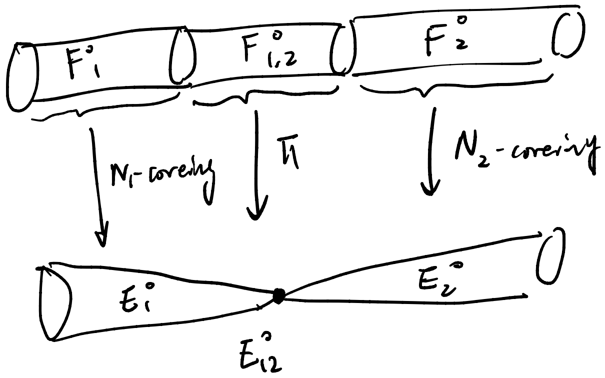


Want m-separating resolution: $\mu : Y \to \mathbb{C}^n$

with exceptional divisors $E_1, \dots, E_k$, $N_i = \mathrm{ord}_f(E_i)$

s.t. $E_i \cap E_j \neq \phi \implies N_i + N_j > m$

Symplectic structure



In a neighborhood away from F_{12}° can pull back ω_E.

but this would degenerate near the boundary.

Consider the case $E_1^\circ, E_2^\circ = \mathbb{C}$

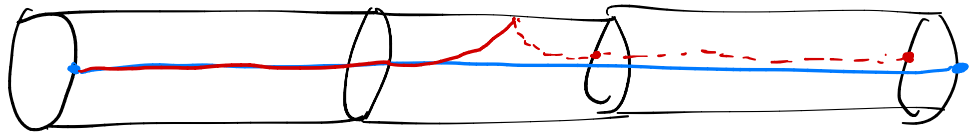
We want to modify $\pi^* dr\, d\theta$ so that it extends to F_{12}°

Instead of using (r, θ)
Can construct coordinates (ν, θ) on F

$\pi^* dr\, d\theta + \varepsilon\, d\nu\, d\theta$ is a symplectic form

Then F is a Liouville domain
and monodromy φ is a symplectomorphism.

φ:



Fixed locus of φ^m, $B_i = F_i$ s.t. $N_i \mid m$

<u>Thm</u> (McLean)

$\exists$ spectral sequence

$$E_1^{p,q} = \bigoplus_{\iota(i)=p} H_{n-(p+q)-CZ(\varphi^m, F_i)}(B_i, \mathbb{Z}) \Rightarrow HF_*(\varphi^m, +)$$

for some function ι.

In our case the spectral sequence degenerates at first page.

In curve case a deformation is needed to get degeneration.

A general method is not known.

By Lefschetz duality,

$$HF_*(\varphi^m, +) \cong \bigoplus H^{*+n-1+CZ}(B_i, \partial B_i)$$

$$\cong \bigoplus H_c^{*+n-1+CZ}(B_i - \partial B_i)$$

Contact loci

$$\mathcal{X}_m = \{ \gamma \in \mathcal{L}_m(\mathbb{C}^n) \mid \gamma(0)=0,\ f(\gamma(t)) = t^m \bmod t^{m+1} \}.$$

Compute cohomology using filtration

$$F_p = \{ \gamma \in \mathcal{X}_m \mid \mathrm{ord}_{\mathbb{C}^n} \gamma \geq P \}, \quad P = -p$$

$$F_{(p)} = F_p - F_{p+1}.$$

get spectral sequence

$$E_1^{p,q} = H_c^{p+q}(F_{(p)}\mathcal{X}_m) \Rightarrow H_c^{p+q}(\mathcal{X}_m(f))$$

Compute $F_{(p)}$ and get

<u>Prop</u> $$F_{(p)} = \begin{cases} X_{m-dp} \times \mathbb{C}^{np(d-1)} & \text{for } 1 \leq p < \frac{m}{d} \\ M_f \times \mathbb{C}^{np(d-1)} & \text{if } d \mid m \text{ and } p = \frac{m}{d}. \\ \phi & \text{otherwise}. \end{cases}$$

where X_{m-dp} is a fibration over $CS^\circ = X - \text{origin}$.

Hence $$H_c^*(F_{(p)}) = \begin{cases} H^{*-2s}(CS^\circ)(-s) \\ H^{*-2s}(M_f)(-s) \\ 0 \end{cases}$$ by Leray spectral sequence

s depends on m, n, d, p.

By hard Lefschetz and Poincaré duality,

$$H_c^1(CS^\circ) \cong H^0(S)$$
$$H_c^{n-1}(CS^\circ) \cong H^{n-2}_{prim}(S)$$
$$H_c^n(CS^\circ) \cong H^{n-2}_{prim}(S)(-1)$$
$$H_c^{2n-2}(CS^\circ) \cong H^{2n-4}(S)(-1)$$

$$H_c^{n-1}(M_f) = \mathbb{Z}^{\mu(f)}(n-1)$$
$$H_c^{2n-2}(M_f) = \mathbb{Z}(2n-2)$$
$$H_c^{\text{everything else}} = 0.$$

Comparing Hodge weights, get differentials on E trivial except some special cases which can be solved by converting to Borel-Moore homology and computing on level of cycles.

Get

$$H_c^*(\mathcal{X}_m, \mathbb{Q}) = \bigoplus H_c^*(F_{(p)}, \mathbb{Q})$$

Finally, need

$$H_c^{*+n-1+cz}(B_i - \partial B_i) = H_c^{*+(n-1)(2m+1)}(F_{(p)}, \mathbb{Q})$$

<u>Prop</u>

$$\begin{tikzcd}
B_i - \partial B_i \cong \tilde{E}_i^\circ \arrow[rr, "\substack{\text{canonical} \\ N_i\text{-cover}}"] & & E_i^\circ \\
F_{(p)} \arrow[r, "\substack{\mathbb{C}^{m(n-1)-cz} \\ \text{fibration}}"] & \tilde{E}_i^\circ \arrow[ur] &
\end{tikzcd}$$

shift is exactly $(2m+1)(n-1)$.