Deformations of complex and Kähler structures

Defn A Calabi-Yau manifold is a compact Kähler manifold X with $\omega_X \cong \mathcal{O}_X$.

Sometimes there will be assumptions $H^i(\mathcal{O}_X)=0$ for $i=1$, $i=1,2,3$ or just $i>0$.
In deformation theory of moduli spaces, these conditions can simplify calculations.
When X is toric $H^i(\mathcal{O}_X)=0$ $\forall i>0$. also get nice identities by Serre duality.

Complex structures

X cpt $\mathbb{C}$-mfd. Consider $\mathbb{C}$-str on X compatible with sm. str.

Teichmüller space: $\mathrm{Teich}(X)$

$\{\mathbb{C}\text{-str on } X\}/\sim_0$

where $J \sim_0 J'$ if $\exists$ diffeo φ isotopic to id $J=\varphi^* J'$.

moduli sp. of $\mathbb{C}$-str: $\mathcal{M}_{cx}(X)$

$\{\mathbb{C}\text{-str on } X\}/\sim$

where $J\sim J'$ if $\exists \varphi$, $J=\varphi^* J'$.

$\mathcal{M}_{cx}$ is in general not a variety. but we can study locally.

how can we deform J to get another $\mathbb{C}$-str J'?

A deformation of X is a smooth proper morphism with $0\in S$

$Y\to S$ s.t. fiber over 0 is $Y_0=X$.

say Y is a family of $\mathbb{C}$-mfds parametrized by S.

think of "small perturbation" as family over $S=\mathrm{Spec}\,\mathbb{C}[x]/(x^2)$.

A universal family $\mathrm{Def}(X)$ would parametrize all deformations of X where $0\in \mathrm{Def}(X)$ is the trivial deformation so it has a universal deformation

$$\mathcal{X} \longrightarrow \mathrm{Def}(X),$$

Idea: if $s \in \mathrm{Def}(X)$ is "some deformation X'", then $Y_s = X'$.

For $Y' \to S$, we have
$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ S \arrow[r] & \mathrm{Def}(X) \end{tikzcd}.$$

Thm If $H^0(X, T_X) = 0$ then $\exists$ universal deformation

If S is contractible (say a small disk) then we expect $Y \to S$ to trivialize so $Y = X \times S$ and each fiber (Y_s, J) is diffeo to X, say $\varphi_s : X \cong Y_s$ by and φ_s is continuous in s. so $X' = (X, \varphi_s^* J)$ is a "deformed" $\mathbb{C}$-str.

How do we "measure" the deformation?

The idea is to measure how much the complex part changed.

We have $T_X = T_X^{1,0} \oplus T_X^{0,1}$ decomposed into z and $\bar{z}$ parts

$\Omega_X = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$.

As J vary continuously the $(1,0)$ forms also vary continuously.

if X' is close to X, and then

$$\pi_X^{1,0}|_{\Omega_{X'}^{1,0}} : \Omega_{X'}^{1,0} \longrightarrow \Omega_X^{1,0}$$

is an isomorphism.

The difference between J and J' is then

$$s : \Omega_X^{1,0} \longrightarrow \Omega_X^{0,1}$$

$$s = -\pi_X^{0,1} \circ \left(\pi_X^{1,0}|_{\Omega_{X'}^{1,0}}\right)^{-1}$$

the image of s is $\pi_X^{0,1}(\Omega_{X'}^{1,0})$.

so when $X = X'$, $s = 0$.

Conversely, a small map $s \in \Gamma(X, T_X^{1,0} \otimes \Omega_X^{0,1})$ induces an alm. $\mathbb{C}$-str.

Let $\Omega_{X'}^{1,0}$ be the graph of s. $\Omega_{X'}^{0,1}$ is the J-conjugate of $\Omega_{X'}^{1,0}$.

Then the decomposition $\Omega_{X'}^{1,0} \oplus \Omega_{X'}^{0,1}$ gives J'.

How to tell if J' is integrable?

$T_X^{1,0} \otimes \Omega_X^{0,*}$ has a Lie bracket

$$[\alpha \otimes d\bar{z}_I, \alpha' \otimes d\bar{z}_J] = [\alpha, \alpha'] \otimes (d\bar{z}_I \wedge d\bar{z}_J)$$

and differential $\bar{\partial}$ acting by $\Omega_X^{0,*} \to \Omega_X^{0,*+1}$

(differential graded Lie algebra)

<u>Prop</u> J' is integrable iff $\bar{\partial} s + \frac{1}{2}[s,s] = 0$.

<u>Rmk</u> Recall Cauchy-Riemann is $\bar{\partial} f = 0$.

A holo fcn w.r.t J' satisfies $\bar{\partial} f + s \cdot \partial f = 0$. $(\bar{\partial}_J \cdot f = 0)$

Consider a curve $(-\varepsilon, \varepsilon) \longrightarrow \mathrm{Def}(X)$

$(*)$ this corresponds to $s(t)$ s.t. $\bar{\partial} s(t) + \frac{1}{2}[s(t), s(t)] = 0$. $s(0) = 0$.

so $s'(0)$ is a "tangent vector of $\mathrm{Def}(X)$".

To get the tangent of Teichmüller space, we need to divide by $\mathrm{Diff}(X)_0$.

The action by $\mathrm{Diff}(X)_0$ on $\mathbb{C}$-str is the same as the action of a family $v(t) \in \Gamma(X, T_X^{1,0})$ with flow $\varphi_t : X \to X$, $\varphi_0 = \mathrm{id}$.

Expand s as a power series $s = \sum s_i t^i$, $s_i \in \Gamma(X, T_X^{1,0} \otimes \Omega_X^{0,1})$

using the condition $(*)$ what do we know about s_1?

$(*)$ mod t^2 gives $\bar{\partial} s_1 = 0$. but we still need to mod out $\mathrm{Diff}(X)_0$

Write $s = \sum s_{ij} \partial z_i \otimes d\bar{z}_j$ locally, so

$$dz_i - s(dz_i) = dz_i - \sum s_{1,ij} \, t \, d\bar{z}_j + O(t^2)$$

Write $v(t) = \sum f_i \partial z_i$ then $\varphi_t(z_i) = z_i + t f_i + O(t^2)$

so $\varphi_t(dz_i - s(dz_i)) = dz_i + t\left(\sum(-s_{1,ij} + \frac{\partial f_i}{\partial \bar{z}_j}) d\bar{z}_j \right) + t \sum \frac{\partial f_i}{\partial z_j} \partial z_j + O(t^2)$.

There are (1,0)-forms of X', spanning

$$dz_i - t\Big(\sum \underbrace{(s_{1,ij} - \tfrac{\partial f_i}{\partial \bar{z}_j})d\bar{z}_j}_{\bar\partial v.}\Big) \quad \text{as } f_i \text{ vary}$$

Here s_1 can take values in

$$\text{“} Def_1(X) \text{”} = \frac{\ker(\bar\partial : \Omega^{0,1}(T_X^{1,0}) \to \Omega^{0,2}(T_X^{1,0}))}{\operatorname{Im}(\bar\partial : C^\infty(T_X^{1,0}) \to \Omega^{0,1}(T_X^{1,0}))}$$

$= D_X(\mathbb{C}[t]/(t^2))$

$$= H^1_{\text{Dolbeault}}(X, \Theta_X) \quad \text{first order infinitesimal deformation}$$

This gives a map $T_0 S \to H^1_{\bar\partial}(X, \Theta_X)$ Kodaira-Spencer map.

<u>AG explanation</u> (X,J) is given by charts U_i, and holo trans fun φ_{ij}

perturbing φ_{ij} yields v.f. v_{ij} satisfying Čech-cocycle condition

mod out biholo $\psi_i : U_i \xrightarrow{\sim} U_i$ is modding out Čech coboundary.

so also get $H^1(X, \Theta_X)$

Let A Artin local $\mathbb{C}$-alg e.g. $\mathbb{C}[t]/(t^2)$. with unique maximal ideal m_A.

Let $$D_X(A) = \frac{\{s \in \Gamma(X, T_X^{1,0} \otimes \Omega_X^{0,1}) \otimes m_A \mid \bar\partial s + \frac{1}{2}[s,s] = 0\}}{\Gamma(X, T_X^{1,0}) \otimes m_A}$$

$s \in D_X(A)$ corresponds to $Y_A \to \operatorname{Spec} A$ (inf def of $\mathbb{C}$-str over A).

But what about converse, given s_1 is there $s(t)$ with $s'(0) = s_1$?

$$\bar\partial s(t) + \tfrac{1}{2}[s(t), s(t)] = 0 \implies \bar\partial s_2 + \tfrac{1}{2}[s_1, s_1] = 0,\ \bar\partial s_3 + \cdots$$

so $[s_1, s_1] \in \operatorname{Im}\bar\partial$. Recall $[s_1, s_1] \in \ker\bar\partial$ (wedge of closed form is closed)

so s exists only if $[s_1, s_1] = 0$ in $H^2(X, T_X)$

This is the primary obstruction to s.

If $[s_1, s_1] = 0$, then the next obs is $[s_1, s_2] \in H^2(X, T_X)$. etc.

If $H^2(X, TX) = 0$ then there is no obstruction, and s always extends.

But this is not true for CY mfd,

so the following is surprising.

Thm (Bogomolov - Tian - Todorov) true if $H^1(X, \mathcal{O}_X) = 0$.

X cpt C.Y with $H^0(X, TX) = 0$ (automorphisms are discrete)

Teich(X) is smooth near X with tangent $H^1(X, TX)$. (no obs)

$\mathcal{M}_{cx}(X)$ is an orbifold near X with tangent $H^1(X, TX)$.

Proof of no obs is by showing an element $D_X(A_n)$ lifts to $D_X(A_{n+1})$

for $A_n = \mathbb{C}[t]/(t^n)$.

Čech argument can show this lifts only if an obs class in $H^2(X, T_X)$ vanishes

Kahler deformations

- the space of Kahler forms $[\omega] \in H^{1,1}(X)$ form open convex cone.
- Deformations are constrained by Hodge symmetry.
- X and mirror $\check{X}$ have mirror map

  $\mathbb{C}$-moduli of $X \longleftrightarrow$ Kahler moduli of $\check{X}$.