NOTES FOR RENNES

Abstract. This is the notes taken from talks given by Anne Moreau, Mircea Mustaţă, and Enrica Floris at the Equivariant Methods at the Henri Lebesgue Center in June 2025.

Contents

1. Vertex algebra and singularities

Vertex algebra is a complicated algebraic structure used in the proof of the Moonshine conjecture in constructing infinite dimensional representations of sporadic groups. It is also used in representation theory of Kac-Moody algebras and infinite dimensional Lie algebras. In physics, this is used in 2D conformal filed theory.

Given a vertext algebra, one can take its automorphism groups which is an algebraic group. It character is a modular function. Its associated variety is an affine (Poisson) algebraic variety, and its Lie algebra is a Lie algebra.

1.1. **Motivation: Moonshine conjecture.** The Moonshine conjecture is an unexpected connection between the *Monster group* M – the largest simple sporadic group – and the modular function j – a holomorphic function on the upper half plane. McKay observed that the coefficients of the Laurent series of j relates to the dimensions of the irreducible representations of M. The Moonshine conjecture states that there exists a infinite dimensional representation V of M such that $V = \oplus_{n\geq 0} V_n$ where

$$\mathrm{ch}V(q) := \sum_{n\geq 0} q^{n-1} = j(\tau) - 744, \quad q = e^{2\pi i\tau}.$$

Such a representation is constructed with a vertex algebra structure where the automorphism group of V is M and the character of V is j.

1.2. **Vertex algebra.** A *vertex algebra* is

- A complex vector space V,
- equipped with a *vacuum vector* $|0\rangle$,
- for $a \in V$, a collection of *fields* $a_{(n)} \in \mathrm{End}\, V$ for $n \in \mathbb{Z}$ such that

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}\, V[\![z^{\pm}]\!],$$

- a *translation operator* $T \in \mathrm{End}\, V$,

satisfying the axioms

- $V \to \mathrm{End}\, V[\![z^{\pm}]\!]$ linear and $a(z)b \in V(\!(z)\!)$ for each $b \in V$,
- $|0\rangle(z) = \mathrm{Id}$, and $a(z)|0\rangle \in V[\![z]\!]$ with $a(0)|0\rangle = a$. This means the field map is injective.
- $Ta(z) = [T, a(z)] = \partial_z a(z)$, so that $(Ta)_{(n)} = -na_{(n-1)}$ and T is uniquely determined.
- *locality:* there exists some $N = N_{a,b}$ such that

$$(z-w)^N a(z)b(w) = (z-w)^N b(w)a(z) \in \mathrm{End}\, V[\![z^{\pm}, w^{\pm}]\!].$$

The locality condition is equivalent to several useful identities

- $$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1)}} + :a(z)b(w):$$

  where the first sum has poles in $(z-w)$ with $c_j(w) \in \mathrm{End}\, V[\![w^{\pm}]\!]$ and $c_j(w)v \in V(\!(w)\!)$ for $v \in V$. The sum makes sense over $|z| > |w|$. The term $:a(z)b(w):$ is a type of product of series that has no poles.

- $$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{n}{j} (a_{(j)}b)_{(m+n-j)},$$

$$(a_{(m)}b)_{(n)} = \sum_{i\geq 0} \binom{m}{i} (a_{(m-i)}b_{(n+i)} - (-1)^m b_{(m+n-i)}a_{(i)})$$

Example 1.1 (Heisenberg algebra)**.** Let $H = \mathbb{C}[x_1, x_2, \dots]$ and set

$$h_{(n)}(p) := \begin{cases} x_n p & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ n\frac{\partial}{\partial x_n} p & \text{if } n > 0. \end{cases}$$

We get $[h_{(m)}, h_{(n)}] = m\delta_{m,-n}\,\mathrm{Id}$. This means H is a representation of the Lie algebra

$$\mathcal{H} = \mathbb{C}[t^{\pm}] \oplus \mathbb{C}\cdot \mathbf{1}$$

where $[t^m, t^n] = m\delta_{m,-n}\mathbf{1}$ and $\mathbf{1}$ is central, sending t^n to $h_{(n)}$ and $\mathbf{1}$ to Id.

Note that although the definition requires a field for each element of H, we only gave some unspecified field $h(z)$. This still defines a vertex algebra as we will see later.

To finish defining the vertex algebra structure, we define $|0\rangle = 1$ and note $[h(z), h(w)] = \partial_w \delta(z, w)$ where $\delta(z, w) = \sum w^n z^{-n-1}$. Locality is then satisfied due to the identity

$$(z - w)^{j+1} \partial_w^j \delta(z - w) = 0$$

◇

Definition 1.2. Let V be a vector space. A series $a(z) \in \operatorname{End} V[\![z^{\pm}]\!]$ is a *field* if $a(z)b \in V(\!(z)\!)$ for any $b \in V$. Two fields a, b are *mutually local* if there exists some N such that

$$(z - w)^N [a(z), b(w)] = 0.$$

Lemma 1.3. *If a, b, c are mutually local, then $\partial_z a(z), b(z)$ are mutually local and $: a(z)b(z) :$ and $c(z)$ are mutually local.*

We can give a new definition of vertex algebra:

- A complex vector space V,
- a vacuum vector $|0\rangle$,
- a translation operator T,
- a set of mutually local fields $\mathcal{S}$,

satisfying the axioms

- $T|0\rangle = 0$,
- $[T, a(z)] = \partial_z a(z)$,
- V is spanned by $a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)}|0\rangle$ for $a^{i_j} \in \mathcal{S}$.

The equivalence of the two definitions is given by the linear isomorphism that sends $a(z) \in \langle \mathcal{S} \rangle_V$ to $a(z)|0\rangle \in V$ where $\langle \mathcal{S} \rangle_V$ is the space generated by $\mathcal{S}$.

Example 1.4 (The Virasoro vertex algebra)**.** Let H be a before. Take $L(z) = \frac{1}{2} : h(z)h(z) :$ and write $L(z) = \sum L_m z^{-m-2}$, so $L_m = L_{(m+1)}$. Then

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} \operatorname{Id}.$$

We set $Vir := \langle L(z) \rangle_H$.

◇

1.3. **Characters of vertex algebras.** Let V be a vertex algebra, $V = \oplus_{a \in \mathbb{Z}} V_n$ be a grading. We say V is *conformal* if there exists $w \in V$, with $w(z) = \sum_{m \in \mathbb{Z}} L_m z^{-n-2}$ such that

- $L_{-1} = w_{(0)} = T$,
- $L_0 = w_{(1)}$ acts semi-simply on V, with V_n the eigenspace $\ker(L_0 - n \operatorname{Id})$,
- $[L_m, L_n] = \dots + \frac{m^3 - m}{12} \delta_{m,-n} \cdot c \cdot \operatorname{Id}$ for some $c \in \mathbb{R}$ called the *central charge*.

The character of V is

$$\chi_V(q) := \operatorname{tr}_V(q^{L_0 - c/24}) = q^{-c/24} \sum_{n \in \mathbb{Z}} \dim V_n q^n.$$

Example 1.5. The Heisenberg vertex algebra H has central charge 1, where L_0 acts on H gives $\deg x_n = n$, and

$$\chi_H(q) = q^{-1/24} \prod_{n \geq 0} (1 - q^n)^{-1}.$$

◇

1.4. **Associated variety.** Say $V = \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)}|0\rangle | n_i \in \mathbb{Z}\}$. Note that

$$a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)}|0\rangle(z) = \frac{1}{n_1! \dots n_r!} : \partial_z^{n_1} a^{i_1}(z) \dots \partial_z^{n_r} a^{i_r}(z) : .$$

We set $c_2(V) := \mathrm{span}\{a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)}|0\rangle | n_1 + \dots + n_r \geq 1\} = \mathrm{span}\{a_{(-2)}b | a, b \in V\}$. The quotient $V/c_2(V) =: R_V$ is a commutative, associative, Poisson algebra with $a \cdot b := a_{(-1)}b$ and $\{a, b\} := a_{(0)}b$. Assume R_V is finitely generated, meaning $\mathcal{S}$ is finite, we take

$$\widetilde{X}_V := \mathrm{Spec}\, R_V, \quad X_V := (\widetilde{X}_V)_{\mathrm{red}}$$

to be the *character variety* of V.

Example 1.6. We have $R_H = \mathbb{C}[x_1]$ and $\widetilde{X}_H \cong \mathbb{C}$ with the trivial Poisson structure. $\diamond$

Denote $J\widetilde{X}_V = \mathrm{Spec}\, JR_V$ the arc space of X_V. Here JR_V is the *jet algebra* of R_V, which is a differential graded algebra, and corresponds to a commutative vertex algebra. The *Hilbert polynomial* is

$$HP_{JR_V}(q) := \sum_{n \geq 0} \dim(JR_V)_n q^n$$

where the grading is given by the order of jets.

Example 1.7. The jet algebra of $\mathbb{C}[x_1, \dots x_n]$ is $\mathbb{C}[\partial^j x_i : j \geq 0, i = 1, \dots, n]$. We have $JR_H = \mathbb{C}[\partial^j x_i]$ and $HP_{JR_H}(q) = \prod_{n \geq 0}(1 - q^n)^{-1}) = q^{1/24}\chi_H(q)$. $\diamond$

1.5. **Virasoro vertex algebra.** Let $\mathscr{L} := \oplus_{n \in \mathbb{Z}} L_n \oplus \mathbb{C}\mathbf{1}$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}\mathbf{1}$ and $\mathbf{1}$ central. Fix $c \in \mathbb{C}$ and take $\mathscr{L}^+ := \oplus_{n \geq -1} L_n$. Define

$$\begin{aligned} Vir^c :=& U(\mathscr{L}) \otimes_{U(\mathscr{L} \oplus \mathbb{C}\mathbf{1}} \mathbb{C}_c \\ \cong& \mathrm{Spec}_{\mathbb{C}}\{L_{-n_1-2}, \dots, L_{-n_r-2}|0\rangle : n_i \geq 0\}. \end{aligned}$$

This has a unique vertex algebra structure such that $|0\rangle = \mathbf{1}, \mathcal{S} = \{L(z)\}$.

Remark 1.8. Vir^c is not always simple as a vertex algebra or $\mathscr{L}$-representation. It is not simple if and only if there exists $u, v \geq 2$ coprime, $c = 1 - 6\frac{(u-v)^2}{uv}$. Set Vir_c the simple quotient of Vir^c. $\diamond$

Example 1.9. $(u, v) = (2, 2r+1)$, then we have Kac-Weil formula

$$\chi_{Vir_{2,2r+1}}(q) = q^{-c/24} \prod_{n \neq 0, \pm 1, 2r+1} (1 - q^n)^{-1}$$

and

$$R_{Vir_{2,2r+1}} = \mathbb{C}[x]/x^r.$$

Also,

$$HP_{JR_{Vir_{u,v}}}(q) = q^{c/24}\chi_{Vir_{u,v}}(q)$$

if and only if $(u, v) = (2, 2r+1)$. $\diamond$

1.6. **Li filtration.** A vertex algebra V admits a filtration whose graded algebra resemble the jet algebra of R_V. Take

$$F^p V = \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(-n_1-1)} \dots a^{i_r}_{(-n_r-1)}|0\rangle : n_1 + \dots + n_r \geq p\}.$$

Then $\mathrm{Gr}V = \oplus F^p V / F^{p+1} V = R_V \oplus \dots$ is a commutative vertex algebra. Let σ be the quotient maps to the graded pieces, then

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(a_{(-1)}b), \quad \partial\sigma_p(a) = \sigma_{p+1}(Ta), \quad TF^pV \subseteq F^{p+1}V$$

and R_V generates $\mathrm{Gr}V$ as a differential algebra.

Example 1.10. $\mathrm{Gr}Vir^c = \mathbb{C}[L_{-2}, L_{-3}, \dots]$ and $R_{Vir^c} = \mathbb{C}[x]$ for $x = L_{-2}|0\rangle = \sigma_0(L_{-2}|0\rangle$. ◇

There exists a surjection of differential Poisson algebras $JR_V \to \mathrm{Gr}V$ because

$$\mathrm{Hom}_{\text{diff alg}}(JR, A) \cong \mathrm{Hom}_{\text{alg}}(R, A).$$

Example 1.11 (Affine vertex algebra)**.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. Set $\hat{\mathfrak{g}} := \mathfrak{g}[t^{\pm}] \oplus \mathbb{C}k$ where $[xt^m, yt^n] = [x,y]t^{m+n} + m(x|y)\delta_{m,-n}K$ where $(x|y)$ is a non degenerate two form.

Example 1.12. Let $\mathfrak{g} = \mathfrak{sl}_n$, then $[x,y] = xy - yx$, $(x|y) = \mathrm{tr}(xy)$. ◇

Take $V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t^{\pm}]) \oplus \mathbb{C}K)} \mathbb{C}_k \cong \mathrm{span}_{\mathbb{C}}\{x^{i_1}_{(-n_1-1)} \cdots x^{i_r}_{(-n_r-1)}|0\rangle\}$ where $\{x^i\}$ is a basis of $\mathfrak{g}$. ◇

Remark 1.13. If $\mathfrak{g} = \mathbb{C}$, then $V^k(\mathfrak{g}) = H$. ◇

One can construct more vertex algebras from $V^k(\mathfrak{g})$ via orbifold/commutant, quotient, $H^*(V^k(\mathfrak{g}))$, etc.

We have $R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*] \cong \mathbb{C}[x^i : i \in I]$. This gives

$$\mathrm{Gr}V^k(\mathfrak{g}) \cong \mathrm{Sym}(t^[- 1]\mathfrak{g}[t^{-1}]) \cong J\mathbb{C}[\mathfrak{g}^*]$$

by $x^i t^{-n-1} \mapsto \partial^n x^i$.

In general, $V^k(\mathfrak{g})$ is not simple, and write $L_k(\mathfrak{g})$ the simple quotient. It is hard to compute $R_{L_k(\mathfrak{g})}$ but some $X_{L_k(\mathfrak{g})}$ is known.

Example 1.14. Let $\mathfrak{g} = \mathfrak{sl}_n$, $k = -n + p/q$, $p \geq n$. Then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotents.

Let $\mathfrak{g} = \mathfrak{sl}_2 = \{(x,y,z) : x^2 + yz = 0\}$ with $k = -2 + 2/3$, $L = L_{-4/3}(\mathfrak{g} = \mathfrak{sl}_2)$. Then $\widetilde{X}_L = \mathbb{C}[x,y,z]/I$, $\sqrt{I} = (x^2 + yz)$ and $\chi_L(q) = HP_{\mathfrak{sl}_2}(q)$.

Let $Z(V)$ be the center of a vertex algebra, which is a commutative vertex algebra. Then $Z(V^k(\mathfrak{g})) = \mathbb{C}$ whenever k is less than $-h^\vee$ the dual Coxeter number. ◇

Definition 1.15. A vertex algebra V is *lisse* if $X_V = \{\mathrm{pt}\}$, which happens if and only if $\dim R_V < \infty$. It is *quasi-lisse* if X_V has finitely many symplectic leaves.

Example 1.16. The chiral differential vertex algebra $\mathcal{D}^{\mathrm{ch}}_X$ is quasi-lisse but not lisse, as $\tilde{X}_{\mathcal{D}^{\mathrm{ch}}_X} \cong T^*X$ has one symplectic leaf. ◇

Theorem 1.17. *If V is quasi-lisse,*

$$SS(V)_{\mathrm{red}} := \mathrm{Spec}\,\mathrm{Gr}V \cong (J\widetilde{X}_V)_{\mathrm{red}}$$

as topological spaces.

Conjecture 1.18. *If V is quasi-lisse, then X_V is reduced (so its closure has one symplectic leaf).*

2. Minimal model program and singularities

2.1. Motivation: MMP for surfaces over $\mathbb{C}$.

Theorem 2.1. *Let S be a smooth surface, $E \subseteq S$ a (-1)-curve. Then there exists $\overline{S}$ such that $\mu : S \to \overline{S}$ is an isomorphism on $S \setminus E$ and $\mu(E) = \mathrm{pt}$.*

Remark 2.2. E is a (-1)-curve is and only if $K_S \cdot E = -1$ and $E^2 = -1$. We have $S \cong \mathrm{Bl}_{\mathrm{pt}}\,\overline{S}$. ◇

Theorem 2.3 (Factorization of birational morphisms)**.** *Let S_1, S_2 be smooth surfaces. If $f : S_1 \to S_2$ is birational, then there exists smooth surfaces $S_1 \cong Y_1, Y_2, \dots, Y_k \cong S_2$ with maps*

$$\pi_i : Y_{i-1} \to Y_i$$

such that $Y_{i-1} = \mathrm{Bl}_{q_i} Y_i$ and $\pi_k \circ \cdots \circ \pi_1 = f$.

Consequently, if S is a smooth surface not containing any (-1)-curve, then $f : S \to S'$ birational and S' smooth means $S \cong S'$. So S is minimal with respect to the order where $S_1 \preceq S_2$ if and only if there exists $f : S_2 \to S_1$ birational.

By contracting all (-1)-curves, one can show minimal surface exists after finitely many steps, as b_2 goes down after contracting a 2-cycle.

Theorem 2.4 (Classification of minimal surfaces)**.** *If S is a minimal surface, then either*

- *K_S is semi-ample if and only if $\phi_{|mK_S|} : S \to Z$ has fiber with trivial canonical bundle,*
- *S is $\mathbb{P}^2$ or $\mathbb{P}_C(\mathcal{E})$ for a rank 2 bundle $\mathcal{E}$ over a curve C.*

In higher dimensions, a condition on self intersection $D \cdot D$ does not imply D is contractible. We instead consider the condition that $K_X \cdot C < 0$ for any $C \subseteq D$. Also, if $f : X \to \overline{X}$, then $\overline{X}$ is not necessarily smooth, so we need to consider singular varieties.

Example 2.5. Let Y be smooth of dimension 3, and $X = \operatorname{Bl}_p Y$. Then Y is a K_X-negative contraction, but the exceptional divisor E satisfies $E^3 = 1$.

If Y is smooth and $C \subseteq Y$ is a smooth curve, then $X = \operatorname{Bl}_C Y$ is a $\mathbb{P}^1$ fibration over C, where $K_X \cdot (\text{fiber}) = -2$.

Consider $\overline{X} = (xy + zw = 0) \subseteq \mathbb{A}^4$ with singularity at 0. The blow up of $\mathbb{A}^4$ resolves $\overline{X}$, and we have $K_X = \pi^* K_{\overline{X}} + E$. ◇

2.2. **Discrepancy.** Fundamentally, the idea used in MMP is to take a resolution of singularity $f : Y \to X$ with $K_Y = f^* K_X + \sum a_i E_i$, and encode the singularities in the coefficients a_i.

Definition 2.6 (Canonical divisor)**.** Let X be normal. Write $\det TX^{sm} = \mathcal{O}(-\sum d_i D_i^\circ)$. The *canonical divisor* is $K_X = \sum d_i D_i$ where $D_i = \overline{D_i^\circ}$.

Remark 2.7. Since $\operatorname{Sing} X$ has codimension at least 2, K_X is defined up to linear equivalence.

K_X is not Cartier nor $\mathbb{Q}$-Cariter in general, so $f^* K_X$ does not make sense.

If $\mu : X_1 \to X_2$ is birational, then $\mu_*(K_{X_1}) = K_{X_2}$. ◇

Definition 2.8. A *pair* (X, D) is the data of a normal variety X and a $\mathbb{Q}$-Weil divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier.

Example 2.9. If X is smooth, then $(X, 0)$ is a pair.

If X is $\mathbb{Q}$-factorial and D is $\mathbb{Q}$-Weil, then $(X < D)$ is a pair. ◇

Definition 2.10. Let X be normal and $D = \sum a_i D_i$ be $\mathbb{Q}$-Weil. Then $\operatorname{Supp} D = \sum D_i$ is/has *simple normal crossing* if each D_i is smooth and for all $z \in X$, there exists a neighbourhood U and coordinates $x_1, \dots x_n$ such that $D_i = (x_i = 0)$ in U.

A *log resolution* of (X, D) is a resolution of singularity $f : Y \to X$ such that E is divisorial, and $E + \widetilde{D}$ has SNC support.

BY Hironaka's work, resolutions and log resolutions exist, and are given by compositions of blow-ups.

Example 2.11. Let $D = (x^m - y^m = 0)$ be the union of m lines. Let $\pi : \operatorname{Bl}_0 \mathbb{A}^2 \to \mathbb{A}^2$ be the blow-up. Then π is a log resolution of $(\mathbb{A}^2, D)$ and $\pi^* D = \widetilde{D} + mE$. ◇

Definition 2.12 (Discrepancies)**.** Let (X, D) be a pair, and $f : Y \to X$ birational. Write $K_Y = f^*(K_X + D) + \sum a_i E_i$ such that the support of E_i is in the exceptional divisor or $\widetilde{D}$. Then $a(E_i, X, D) := a_i$ is the *discrepancy* of E_i.

Remark 2.13. If $E_i \subseteq \operatorname{Supp} \widetilde{D}$, and $\widetilde{D} = \sum d_i E_i$, then $a(E_i, X, D) = -d_i$. ◇

Example 2.14. If X is smooth, and f is the blow up of a closed subscheme $Z \subseteq X$ of codimension c, then $a(E, X, 0) = c - 1$. ⋄

Definition 2.15. The *discrepancy* of the pair (X, D) is $\operatorname{disc}(X, D) = \inf\{a(E, X, D) : E \text{ exceptional}\}$. The *total discrepancy* is $\inf\{a(E, X, D) : E \text{ divisor of } X\}$.

Definition 2.16. A divisor $D = \sum d_i D_i$ is

$$\begin{pmatrix} \textit{terminal} \\ \textit{canonical} \\ \textit{KLT} \\ \textit{PLT} \\ \textit{LC} \end{pmatrix} \iff \operatorname{disc}(X, D) \text{ is } \begin{pmatrix} > 0 \\ \geq 0 \\ > -1 \text{ and } \lfloor d_i \rfloor \\ > -1 \\ \geq -1 \end{pmatrix}$$

We say X with K_X being $\mathbb{Q}$-Cartier is of these properties if $(X, 0)$ is.

Remark 2.17. KLT stands for Kawamata log terminal, PLT stands for purely log terminal, and LC stands for log canonical. ⋄

Lemma 2.18. *A pair (X, D) is klt or lc if and only if there exists a log resolution $f : Y \to X$ such that*

$$a_i > -1 \text{ or } \geq -1$$

respectively.

If $(X, 0)$ is a pair, then it is terminal or canonical if and only if there exists a log resolution such that

$$a_i > 0 \text{ or } \geq 0$$

respectively.

Example 2.19. Let X be smooth and $\sum D_i$ be SNC. Then $(X, \sum d_i D_i)$ is klt if and only if $d_i < 1$, and lc if and only if $d_i \leq 1$.

Let $X = \mathbb{A}^2$ and $D = (x^m - y^m = 0)$, and $Y = \operatorname{Bl}_0 \mathbb{A}^2$. Then $K_Y = \pi^*(K_X + D) - \widetilde{D} + (1 - m)E$. So $a(E, X, D) = 1 - m$ and (X, D) is lc if and only if $m \leq 2$. Similarly, we have $a(E, X, \frac{1}{2}D) = 1 - \frac{m}{2}$ and $(X, \frac{1}{2}D)$ is lc if and only if $m \leq 4$ ⋄

Proof. Suppose E is exceptional over X with respect to $g : Y_1 \to X$. Let $h : \hat{Y} \to X$ be the resolution of the indeterminant of Y_1 and Y. Write

$$K_{\hat{Y}} = h^*(K_X + D) + a\widetilde{E} + E'$$

such that $\widetilde{E} \not\subseteq \operatorname{Supp} E'$. Let $p : \hat{Y} \to Y_1$ be the factoring map and apply p_* to get $a = \operatorname{disc}(E, X, D)$. Write $q : \hat{Y} \to Y$. Then

$$K_{\hat{Y}} = q^*(K_Y) + \sum \beta_i G_i = q^*(f^*(K_X + D) + \sum b_j \overline{E}_j) + \sum \beta_i G_i.$$

So $\widetilde{E}$ appears in $\sum b_j \overline{E}_j + \sum \beta_i G_i$ and for one of the $\overline{E}_j$ or G_i, we have $a = b_j$ or $a = \beta_i$. In the former case, we are done by assumption.

We may assume q is a composition of blow-ups. Say q is a single blow up along some Z. Then $a = \operatorname{codim}_Y Z - 1 + \operatorname{coeff}_{\widetilde{E}}(q^* \sum b_i \overline{E}_i) = \operatorname{codim}_Y Z - 1 + \sum_{Z \subseteq \overline{E}_i} b_i$. Since $\sum \overline{E}_i$ is SNC, Z is contained in at most $\operatorname{codim}_Y Z$ of them and we are done. □

Proposition 2.20 (K-negative contraction preserves terminality)**.** *Let X be a terminal variety and $f : X \to Y$ birational with irreducible exceptional divisor E, such that K_Y is $\mathbb{Q}$-Cartier and for all C contracted, $K_X \cdot C < 0$, then Y is terminal.*

2.3. **Multiplier ideal.**

Definition 2.21. Let X be smooth, $D \geq 0$ a $\mathbb{Q}$-Weil divisor. The *multiplier ideal* of D is

$$\mathcal{I}(X, D) = \mu_*(\mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor))$$

where $\mu : X' \to X$ is a log resolution of (X, D). By Hartog's lemma $\mathcal{I}(X, D)$ is an ideal sheaf.

Lemma 2.22. *Let X be smooth and $D \geq 0$. Then (X, D) is klt if and only if $\mathcal{I}(X, D) = \mathcal{O}_X$, and lc if and only if $\mathcal{I}(X, cD) = \mathcal{O}_X$ for all $0 < c < 1$.*

Proof. Write $K_{X'} = \mu^* K_X + \sum a_i E_i$ and $\mu^* D = \widetilde{D} + \sum b_i E_i$. Then

$$K_{X'/X} - \lfloor \mu^* D \rfloor = \sum (a_i - \lfloor b_i \rfloor) E_i - \sum \lfloor d_i \rfloor \widetilde{D}_i.$$

Then $\mathcal{I}(X, D) = \mathcal{O}_X$ if and only if $a_i - \lfloor b_i \rfloor \geq 0$ and $-\lfloor d_i \rfloor \geq 0$, if and only if $a_i - b_i > -1$ and $-\lfloor d_i \rfloor \geq 0$, if and only if (X, D) is klt. □

Definition 2.23. Let (X, Δ) be a pair and $\Delta \geq 0$. The *multiplier ideal* of D with respect to (X, Δ) is

$$\mathcal{I}((X, \Delta), D) = \mu_* \mathcal{I}(K_{X'} - \lfloor \mu^*(K_X + \Delta + D) \rfloor)$$

where μ is a log resolution of $(X, \Delta + D)$.

Example 2.24. Let $X \subseteq \mathbb{C}^{n+1}$ be a hypersurface and $X^{\mathrm{Sing}} = 0$ such that $\mathrm{Bl}_0 X$ is a log resolution of X. Say $X = (x_1^d + \cdots + x_{n+1}^d = 0)$. Then

$$K_{\mathrm{Bl}_0 X} = \pi^* K_X + (n - d)E$$

so $I(X, 0) = (\mathfrak{m}_0)^{d-n}$ for $d \geq n$. ◇

2.3.1. *Analytic construction.*

Definition 2.25. A map $\varphi : U \subseteq \mathbb{C}^n \to [-\infty, \infty)$ is *plurisubharmonic* if it is upper semicontinuous and for all line $L \subseteq U$ and $a \in L \cap U$, we have

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(\alpha + re^{i\theta}) d\theta.$$

Example 2.26. If $\varphi(z) = \log |z|$ or $\varphi(z_1, \ldots, z_n) = \log \sum |z_i|$, then φ is plurisubharmonic. ◇

Definition 2.27. If φ is PSH, then the *multiplier ideal* of φ is

$$\mathcal{I}(\varphi)(U) = \{f : U \to \mathbb{C} \text{ holomorphic, measurable} : |f|^2 e^{2\varphi} \in L^1_{\mathrm{loc}}(U).\}$$

Example 2.28. Let $D = \sum a_i D_i, a_i \in \mathbb{Q}_{>0}$ and $U \subseteq X$ such that $D_i \cap U = (y_i = 0)$. Take $\varphi_D := \sum a_i \log |y_i|$. Then

$$\mathcal{I}(\varphi_D)(U) = \{f \in \mathcal{O}_X(U) | \frac{|f|^2}{\prod |y_i|^{2a_i}} \in L^1_{\mathrm{loc}}(U)\}.$$

◇

Theorem 2.29. *Let X be smooth and $D = \sum a_i D_i, a_i \in \mathbb{Q}_{>0}$. Then*

$$\mathcal{I}(X, D)^{\mathrm{an}} = \mathcal{I}(\varphi_D).$$

Proof. Assume D is SNC and take U such that $D_i \cap U = (x_i = 0)$. Let $f \in I(\varphi_D)(U)$ and Assume $f = \prod z_i^{d_i}$. Then

$$\frac{|f|^2}{\prod |x_i|^{2a_i}} = \prod |x_i|^{2|d_i - a_i|} \in L^1_{\mathrm{loc}}(U)$$

if and only if $d_i - a_i \geq -1$, which happens if $d_i \geq \lfloor a_i \rfloor$. hence $f \in \mathcal{O}(-\lfloor D \rfloor)^{\mathrm{an}} = \mathcal{I}(X, D)^{\mathrm{an}}$. □

2.4. **Vanishing theorems.**

Definition 2.30. A $\mathbb{Q}$-Cartier divisor D is *nef* if for all curves $C \subseteq X$, $D \cdot C \geq 0$. It is *big* if $\dim H^0(X, mD) = O(m^{\dim X})$.

Example 2.31. Let $g : Y \to X$ be geometrically finite and A ample on X. Then g^*A is nef and big. ⋄

Theorem 2.32 (Kodaira vanishing theorem)**.** *Let X be smooth projective and A ample Cartier. For all $i > 0$, $H^i(X, K_X + A) = 0$.*

Theorem 2.33 (Vanishing for nef and big)**.** *If B is a nef and big Cartier divisor, then for all I, $H^i(X, K_X + B) = 0$.*

Theorem 2.34 (Kawamata-Viehweg vanishing)**.** *Let N be Cartier and $N \equiv B + \Delta$ is a numerical equivalence. If B is nef and big, $\Delta = \sum a_i \Delta_i$ is SNC and $a_i \in \mathbb{Q} \cap (0, 1)$, then for all $i > 0$, $H^i(X, K_X + N) = 0$.*

Theorem 2.35 (Local vanishing)**.** *Let X be smooth and D be a $\mathbb{Q}$-divisor. If $\mu : X' \to X$ is a log resolution then for all $i > 0$,*

$$R^i\mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor) = 0.$$

Theorem 2.36 (Nadel vanishing)**.** *Let X be smooth projective, D be $\mathbb{Q}$-Weil and L Cartier. If $L - D$ is nef and big, then for all $i > 0$,*

$$H^i(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(X, D)) = 0.$$

We assume the first two, and prove the third.

Proof.

Lemma 2.37. *Let X be smooth and projective, L a divisor on X and m a positive integer. There exists a finite surjection $f : Y \to X$ and a Cartier divisor N on Y such that $f^*L = N^{\otimes m}$.*

Proof. It suffices to prove it for the very ample bundle $\varphi^*\mathcal{O}_{\mathbb{P}^N}(1)$ because of the following. Suppose $L = A_1 \otimes A_2^\vee$ where A_1, A_2 are very ample. Let φ_i be the embeddings induced by A_i. Then if we have maps $f_1 : Y_1 \to X, f_2 : Y_2 \to Y_1$ such that $f_1^* A_1 = N_1^{\otimes m}$ and $f_2^* f_1^* A_2 = N_2^{\otimes m}$, then we have $f_2^* f_1^* L + (f_2^* N_1 \otimes N_2^\vee)^{\otimes m}$.

Let $\nu : \mathbb{P}^N \to \mathbb{P}^N$ be the map $[x_0 : \cdots : x_N] \mapsto [x_0^m : \cdots : x_N^m]$. Then we take $\overline{Y}$ to be $X \times_{\varphi, \mathbb{P}^N, \nu} \mathbb{P}_N$ and Y its normalization. □

Lemma 2.38. *Let X be smooth and $D \subseteq X$ smooth. If $\mathcal{O}_X(D) = \mathscr{L}^{\otimes m}$ for a line bundle $\mathscr{L}$. Let $V(\mathscr{L})$ be the total space of $\mathscr{L}$, and $U \subseteq X$ an open set trivializing $\mathscr{L}$. Say $U \cap D = (y = 0)$ and $V(\mathscr{L}) \cong U \times \mathbb{A}^1$ over U. Then $Y = (y^m - y(x) = 0) \subseteq U \times \mathbb{A}^1$ glue to a smooth Y. The map $\gamma : Y \to X$ is finite of degree m and totally ramified at D. We have*

$$\gamma_*\mathcal{O}_U = \oplus_{i=0}^m \mathscr{L}^{-i}.$$

Lemma 2.39 (Injectivity lemma)**.** *Let $f : Y \to X$ be a finite surjection for X, Y normal. Let $\mathcal{E} \to X$ be a vector bundle. Then for all j,*

$$H^j(X, \mathcal{E}) \to H^j(Y, f^*\mathcal{E})$$

is injective.

We do an induction on r where $\Delta = \sum_{i=1}^r a_i \Delta_i$.

Let $a_1 = \frac{c}{d}$ for $0 < c < d$. Then there exists $p : X' \to X$ of degree d such that X' is smooth and p is generically finite, with $\Delta_1' = p^*\Delta_1 = dA'$ for some A' Cartier on X. We can alo find p such that $p^* \sum_{i=2}^r \Delta_i = \sum \Delta_i'$ is SNC. Set $N' = p^*N$ and $B' = p^*B$.

Then there exists a cyclic map $q : X'' \to X'$ of degree d, totally branched along A, and $N''-cA'' = B'' + \sum_{i=2}^r a_i\Delta_i''$ for $a_i \in (0,1)$.

By induction, $H^j(X'', -N''+cA'') = 0$. Since q is finite, spectral sequences show

$$H^j(X', q_*\mathcal{O}_{X''}(-N''+cA'')) = H^j(X', \oplus\mathcal{O}_{i=0}^{d-1}(-N'+(c-i)A') = 0$$

Since the direct sum contains a copy of $O(-N')$, we conclude $H^j * X', \mathcal{O}_X'(-N')) = 0$, and by injectivity lemma, $H^j(X, -N) = 0$.

□

2.5. Terminal sigularities.

When X is a surface, X is terminal if and only if X is smooth.

Proposition 2.40. *Let X be projective and suppose K_X is $\mathbb{Q}$-Cartier. Then*

- *If $H \subseteq X$ is a general hyperplane section, then $(X,0)$ is terminal/canonical if and only if $(H,0)$ is.*
- *$(X,0)$ is terminal implies* $\operatorname{codim} X \geq 3$.

Proof. Induction on dimension of X. Let $p : X' \to X$ be a resolution of singularity. Let H be a hyperplane section, so $\operatorname{Sing} H \subseteq \operatorname{Sing} X \cap H$. Let H' be the strict transform, so we can assume H' is smooth.

We have $K_{X'}+J' = p^*(K_X+H)+\sum a_iE_i$. If X is terminal, then H is terminal and by induction, $\operatorname{codim}_H \operatorname{Sing} H \geq 3$. Each component of $\operatorname{Sing} H$ is contained in a component of $\operatorname{Sing} X \cap H$, so $\operatorname{codim}_X \operatorname{Sing} X \geq 3$. □

Recall D Cartier is base-point free if for all $x \in X$, there exists $D' \sim D$ effective such that $x \notin \operatorname{Supp} D$. The base locus of D is $\mathrm{Bs}D = \cap_{D'\in|D|}\operatorname{Supp} D'$.

For all D with $|D| \neq \phi$, there exists a resolution of base locus $\mu : Y \to X$ such that $|\mu^*D| = |M| + F$ for $F \geq 0$ and M base-point free, such that $\operatorname{Supp} F = \mathrm{Bs}\mu^*D$.

Theorem 2.41. *If X is smooth and L is Cartier, and* $\mathrm{Bs}L$ *has codimension at least 2 such that for all $x \in X$, there exists $B_x \in |L|$ with (X, B_x) is canonical in a Zariski neighbourhood of x, then for $B \in |L|$ general, we have (X,B) canonical.*

2.6. Inverse of adjunction.

Theorem 2.42 (Singular relative vanishing)**.** *Let (X,D) be a pair and $D \geq 0$. Let $\mu : X' \to X$ be a log resolution. Then for all $j > 0$,*

$$R^j\mu_*\mathcal{O}_{X'}(K_{X'} - \lfloor \mu^*(K_X+D)\rfloor) = 0.$$

Theorem 2.43 (Connectedness of non-klt locus)**.** *Let X be normal and $D = \sum d_iD_i \geq 0$. Let $f : Y \to X$ be a log resolution of (X,D). Set $A = \sum_{a_i>-1} a_iE_i$ and $F = -\sum_{a_i\leq -1} a_iE_i$. Then* $\operatorname{Supp} F$ *is contained in the neighbourhood of every fiber of f.*

Theorem 2.44 (Adjunction for pairs)**.** *Let (X,D) be a pair and S a normal irreducible Cartier divisor. Let $S \not\subseteq \sup B$ and $(X, B+S)$ be a pair. Then*

$$(K_X+B+S)|_S = K_S + B_S$$

for some B_S such that (S,B_S) is a pair.

Theorem 2.45. *If $B \geq 0$, then $(X, S+B)$ is plt in a neighbourhood of S if and only if (S, B_S) is klt.*

3. Singularity and Hodge modules

3.1. **D-modules.** Let X be a smooth irreducible algebraic variety over $\mathbb{C}$ of dimension n. Let $D_X \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ be the subsheaf generated by $\mathcal{O}_X$ and $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$. This is a sheaf of non-commutative ring.

The sheaf D_X admits a filtration $\mathcal{F}_\bullet D_X$ by order of differential operators. If $x_1, \dots, x_n$ are local coordinates, then $\mathcal{F}_p D_X$ is generated by $\mathcal{O}_X \partial_X^\alpha$ for $\alpha_1 + \cdots + \alpha_n \leq p$. The graded pieces

$$\mathrm{Gr}_\bullet^{\mathcal{F}} D_X \cong \mathrm{Sym}^\bullet(\mathscr{T}_X)$$

is a sheaf of commutative rings.

If $\mathcal{M}$ is a coherent D_X-module, then one can take a *good filtration*, i.e. an exhaustive increasing filtration $\mathcal{F}_\bullet \mathcal{M}$ such that $\mathrm{Gr}_\bullet^{\mathcal{F}}(\mathcal{M})$ is locally finitely generated over $\mathrm{Gr}_\bullet^{\mathcal{F}} D_X$, which is a sheaf on the total space of the cotangent bundle. Equivalently, we want $\mathcal{F}_p\mathcal{M}$ to be coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_p \cdot \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$ with equality for $q \gg 0$.

From $\mathcal{M}$ we get its singular support

$$SS(\mathcal{M}) = \mathrm{Supp}(\mathrm{Gr}_\bullet^{\mathcal{F}}\mathcal{M})_{\mathrm{red}} \subseteq T^*X$$

which is independent of $\mathcal{F}_\bullet$.

Theorem 3.1 (Kashiwara)**.** *Every irreducible component of $SS(\mathcal{M})$ has dimension at least n.*

Definition 3.2. Say $\mathcal{M}$ is *holonomic* if $\dim(SS(\mathcal{M})) = n$ or $\mathcal{M} = 0$. These form an abelian category whose objects are of finite length.

3.2. **Riemann Hilbert correspondence.** The category of holonomic D_X-modules with regular singular support is equivalent to the category of perverse sheaves on X.

Saito's Hodge modules are holonomic D-modules with a fixed good filtration and other data and conditions.

Example 3.3.

- $\mathcal{O}_X$ with the filtration

$$\mathcal{F}_p\mathcal{O}_X = \begin{cases} \mathcal{O}_X, p \geq 0 \\ 0, p < 0, \end{cases}$$

  is a holonomic D-module, where $\mathrm{Gr}_\bullet^{\mathcal{F}}(\mathcal{O}_X)$ is the structural sheaf of zero section on T^*X, which has dimension n support. It corresponds to the perverse sheaf $\mathbb{C}_X[n]$.
- Let $U \overset{j}{\hookrightarrow} X$ be open. Then $R^q j_* \mathcal{O}_U$ is roughly $\mathcal{H}^{q+1}_{X \setminus U}(\mathcal{O}_X)$, where $\mathcal{H}^*_Z(-) = R^p\Gamma^\circ_Z(-)$, the local homology of $X \setminus U \to X$. If $U = X \setminus Z$ and $Z = (f = 0)$, then $Rj_*\mathcal{O}_U = \mathcal{O}_X[\frac{1}{f}]$, and

$$\mathcal{H}^1_Z(\mathcal{O}_X) = \mathcal{O}_X[1/f]/\mathcal{O}_X.$$

- Let Z be a smooth dimension r closed subscheme of X. Then $i_+\mathcal{O}_Z \cong \mathcal{H}^{n-r}_Z(\mathcal{O}_X)$.

◇

Holonomic D-modules are perserved by pullbacks and pushforward.

3.3. **b-functions.** Let $0 \neq f \in \mathcal{O}(X)$. Consider $\mathcal{O}_X[1/f, s]f^s$ the free rank 1 module over $\mathcal{O}_X[1/f, s]$ where s is a variable and f^s is a formal generator. This is a D-module via

$$D \cdot f^s = \frac{sD(f)}{f} f^s$$

for $D \in \mathrm{Der}(\mathcal{O}_X)$.

Theorem 3.4 (Bernstein, Kashiwara). *There exists a polynomial $b(s) \neq 0$ such that*

$$b(s)f^s \in D_X[s] \cdot f^{s+1}.$$

The monic generator of ideal of such functions is the b-function $b_f(s)$*.*

Example 3.5.
- Let $Z = (f = 0)$ be smooth. We have local coordinates such that $x_1 = f$. Then $\partial_{x_1} \cdot x_1^{s+1} = (s = 1)x_1^s$ which means $b_f(s)|s + 1$. In fact, $b_f(s) = s + 1$.
- If f is invertible, then $b_f(-1) = 0$ because $b(-1)\frac{1}{f} \in D_X \cdot 1 = \mathcal{O}_X$.
- Let $f = x_1^2 + \cdots + x_n^2$. Then $\partial_{x_i} f^{s+1} = (s+1)2x_i f^s$ and $\partial_{x_i}^2 f^{s+1} = 2(s+1)f^s + s(s+1)4x_i^2 f^{s-1}$. This means $\sum \partial_{x_i}^2 f^{s+1} = (s+1)(2n+4s)f^s$, so $b_f(s)|(s+1)(s+\frac{n}{2})$.

$\diamond$

Remark 3.6. Existence of $b_f(s)$ implies $\mathcal{O}_X[\frac{1}{f}]$ is fintiely generated over D_X. $\diamond$

Remark 3.7. Similar results hold for all sections of $\mathcal{M}_f[s]f^s$ when $\mathcal{M}$ is a holonomic D-module. $\diamond$

Theorem 3.8 (Kashiwara). *All roots of $b_f(s)$ are rational and negative. More precisely, if $\pi : Y \to X$ is a log resolution of (X, Z) and $\pi^*(Z) = \sum a_i E_i$ with $K_{Y/X} = \sum k_i E_i$, then every root of $b_f(s)$ is equal to $-\frac{k_i+l}{a_i}$ for some i and some $l \in \mathbb{Z}_{>0}$.*

Corollary 3.9. *All roots of $b_f(s)$ are at most $-\min \frac{k_i+1}{a_i} = \mathrm{lct}(X, Z)$.*

It is proven by Kollár by definition of $b_f(s)$, integration by parts, and analytic description of lct ($\sup\{c > 0 | \frac{1}{|f|_c^2}$ is locally integrable$\}$) that

$$b_f(-\mathrm{lct}(X, Z)) = 0.$$

Definition 3.10 (Saito). The *minimal exponent* of f is $\tilde{\alpha}(Z) = -$ largest root of $b_f(s)/(s+1) \in \mathbb{Q}_{>0} \cup \{\infty\}$.

By definition, $\mathrm{lct}(X, Z) = \min\{1, \tilde{\alpha}\}$.

Theorem 3.11 (Saito). *$\tilde{\alpha} > 1$ if and only if Z has rational singularity (if and only if (X, Z) is plt).*

3.4. **V-filtrations.** We want to "restrict $\mathcal{O}_X$ to Z", but Z might not be smooth. Let $\iota : X \to X \times \mathbb{A}^1$ be the graph of f. Let $H = (t = 0) \subseteq X \times \mathbb{A}^1$, then we can restrict$\iota_* \mathcal{O}_X$ to H.

Let

$$\begin{aligned} B_f := \iota_+ \mathcal{O}_X &= \mathcal{H}^1_{\iota(X)}(\mathcal{O}_{X \times \mathbb{A}^1}) \\ &= \mathcal{O}_X[t]_{f-t} / \mathcal{O}_X[t] \\ &= \bigoplus_{j \geq 1} \mathcal{O}_X \left[\frac{1}{(f-t)^j} \right] \\ &\overset{\delta = \frac{1}{f-t}}{=} \bigoplus_{j \geq 0} \mathcal{O}_X \partial_t^i \delta. \end{aligned}$$

The V-*fibration* on B_f is a decresing exhaustive filtration by D_X-submodules $(V^d B_f)_{\alpha \in \mathbb{Q}}$ that is discrete and left continuous, i.e. there exists $l \in \mathbb{Z}_{>0}$ such that V^α is concentrated on $(\frac{i-1}{l}, \frac{i}{l}, i \in \mathbb{Z})$, such that

- $t \cdot V^\alpha \subseteq V^{\alpha+1}$ when equality if $\alpha > 0$,
- $\partial_t \cdot V^\alpha \subseteq V^{\alpha-1}$,
- $\partial_t t - \alpha$ acts nilpotently on Gr_V^α,
- each V^α is finitely generated over $D_X\langle t, \partial_t t \rangle$.

The above definition uniquely characterizes $V^\bullet B_f$. From existence of b-functions and rationality of roots, one can show there exists a V-filtration.

3.5. **Connection between V-filtration and b-funftion.** Let $B_f^+ := \iota_* \mathcal{O}_X[\frac{1}{f}] \cong \oplus \mathcal{O}_X[\frac{1}{f}]\partial_t^j \delta$ which contains B_f. There eixsts an isomorphism

$$\iota_* \mathcal{O}_X[\frac{1}{f}] \cong \mathcal{O}_X[\frac{1}{f}, s] f^s$$

such that δ is mapped to f^s, $\partial_t t$-action to the s-action and t-action to the automorphism induced by $s \mapsto s+1$.

The existence of b-function gives

$$b_f(-\partial_t t) \cdot \delta \in D_X[s] \cdot t\delta$$

and rearranging gives the existence of V-filtrations.

We can characterize $V^\bullet B_f$ by the b-function: for any $U \in B_f$, thereexists a monic b_u of minimal degree such that $b_u(s) \cdot u \in D_X\langle s, t\rangle tu$. If $u = \delta$ then $b_u = b_f$.

Theorem 3.12.

$$V^\alpha B_f = \{u \in B_f | \text{ all roots of } b_u \text{ is} \leq -\alpha\}.$$

Example 3.13. We have $\delta \in V^\alpha B_f$ if and only if $\alpha \in \mathrm{lct}(X, Z)$. ⋄

More generally,

Theorem 3.14 (Budur-Saito)**.** *For all α,*

$$\{g \in \mathcal{O}_X | g\delta \in V^\alpha B_f\} = \mathcal{I}(X, (\alpha - \varepsilon)Z)$$

for $0 < \varepsilon \ll 1$.

Similarly we know if $p \in \mathbb{Z}_{\geq 0}, \alpha \in (0,1]$, then $\tilde{\alpha}(Z) \geq qp + \alpha$ if and only if $\partial_t^p \delta \in V^\alpha B_f$.

For all $p \in Z$, with $P \subseteq U \subseteq V$ open, we have *local minimal exponents* from $b_{f|_V}(s) | b_{f|_U}(s)$, and for U small, this converges to $\tilde{\alpha}_p(z)$ and $b_{f,p}(s)$.

3.6. **The Hodge filtration on $\mathcal{O}_X[\frac{1}{f}]$.** Let $p \in X$ have local coordinates $x_1, \ldots, x_n$ and take $w_1, \ldots, w_n \in \mathbb{Q}_{>0}$ with $\deg x_I = w_i$. Let $f = \sum$ monomial of degree 1. If f has an isolated singularity at p, the nwe can compute $b_{f,p}(s)$ and the V-filtration.

A pure Hodge module is understood as a family of pure Hodge structures with singularities. A mixed Hodge module is a homological formalism set up for Hodge theory.

The data of a *mixed Hodge module* consists of $(\mathcal{M}, \mathcal{F}_\bullet \mathcal{M}, \mathcal{P}, \alpha, W_\bullet)$ and conditions defined inductively on dimension of support by vanishing and nearby cycles, such that in dimension zero, these are mixed Hodge structures.

- $\mathcal{M}$ is a holonomic D_X-module.
- $\mathcal{F}_\bullet \mathcal{M}$ is a good filtration called the Hodge filtration.
- $\mathcal{P}$ is a perverse sheaf over $\mathbb{Q}$
- $\alpha : \mathcal{P}_\mathbb{C} \to \mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$ is an isomorphism.
- $W_\bullet$ is a finite increasing filtration called the weight filtration.

There exists an abelian category of MHM on X, with morphisms strict with respect to $\mathcal{F}_\bullet$ and $W_\bullet$. The categories $D^b(MHM(X))$ satisfies six-functor formalism.

Example 3.15. $\mathbb{Q}_X^H[n] := (\mathcal{O}_X, \mathcal{F}_\bullet \mathcal{O}_X, \mathbb{Q}_X[n], \text{weight} = n)$ is a MHM. One usually study the MHM given by functors applied to $\mathbb{Q}_X^H[n]$. ⋄

Remark 3.16. We may allow X to be singular by embedding (locally) into smooth schemes. For each Z, $\mathbb{Q}_Z^H = \pi_Z^* \mathbb{Q}_{\mathrm{pt}}^H$ where $\pi_Z : Z \to \mathrm{pt}$.

All HM we consider are polarizable. ⋄

3.7. **Graded de Rham complex.** If $\mathcal{M}$ is a D_X-module, then

$$\mathrm{DR}_X(\mathcal{M}) = 0 \to \mathcal{M} \to \Omega^1_X \otimes \mathcal{M} \to \ldots \to \Omega^n_X \otimes \mathcal{M} \to 0$$

is a complex over $\mathbb{C}$ concentrated from degree $-n$ to 0. If $\mathcal{F}_\bullet\mathcal{M}$ is a good filtration, then we get a filtration on $\mathrm{DR}_X(\mathcal{M})$ with graded pieces

$$\mathrm{Gr}^{\mathcal{F}}_p \mathrm{DR}_X(\mathcal{M}) = 0 \to \mathrm{Gr}^{\mathcal{F}}_p(\mathcal{M}) \to \Omega^1_X \otimes \mathrm{Gr}^{\mathcal{F}}_{p+1}(\mathcal{M}) \to \ldots \to \Omega^n_X \otimes \mathrm{Gr}^{\mathcal{F}}_{p+n}(\mathcal{M}) \to 0.$$

This gives $\mathrm{Gr}^{\mathcal{F}}_{-p}\mathbb{Q}^H_X[n] = \Omega^p_X[n-p]$.

Since every morphism of MHM is strict with respect to the Hodge filtration, we get

$$D^b(MHM(X)) \to D^b_{\mathrm{Coh}}(X)$$

commutes with proper pushforward and duality.

3.8. **Vanishing and nearby cycles.** Let $Z = (f = 0)$ for $0 \neq f \in \mathcal{O}(X)$. We have a V-filtration on $B_f = \iota_*\mathcal{O}_X = \oplus_{j\geq 0}\mathcal{O}_X\partial_t^j\delta$. This carries a Hodge filtration $\mathcal{F}_{p+1}B_f = \oplus_{j\leq p}\mathcal{O}_X\partial_t^j\delta$ which induces filtrations on $\mathrm{Gr}^\alpha_V B_f$ as a filtered D_X-module.

The *nearby cycle* is

$$\psi_f(\mathcal{O}_X) := \bigoplus_{\alpha\in(0,1]} \mathrm{Gr}^\alpha_V B_f$$

as a filtered D_X-module, and the *vanishing cycle* is

$$\Psi_f(\mathcal{O}_X) = \mathrm{Gr}^0_V B_f(-1) \oplus \bigoplus_{\alpha\in(0,1)} \mathrm{Gr}^\alpha_V B_f.$$

The MHM corresponding to these filtered D-modules are $\psi_f\mathbb{Q}^H_X[n], \Psi_f\mathbb{Q}^H_X[n]$ have rational structures given by Deligne's nearby/vanishing cycles.

If $Z \overset{i}{\hookrightarrow} X$ is closed and $U = X \setminus Z \overset{j}{\to} X$, we have an exact triangle

$$i_*i^!\mathbb{Q}^H_X[n] \to \mathbb{Q}^H_X[n] \to j_*\mathbb{Q}^H_U[n] \to^{+1}.$$

As $\mathcal{O}_X$-modules, $\mathcal{H}^q(J_*\mathbb{Q}^H_U[n]) = R^qj_*\mathbb{Q}^H_U[n]$ and $\mathcal{H}^q_Z(i_*i^!\mathbb{Q}^H_X[n] = \mathcal{H}^q_Z(\mathcal{O}_X)$.

Let Z be a reduced hypersurface, then the above becomes

$$0 \to \mathcal{O}_X \to \mathcal{O}_X[\frac{1}{f}] \to \mathcal{H}^1_Z(\mathcal{O}_X) \to 0.$$

Thus understanding Hodge filtrations on $\mathcal{O}_X[\frac{1}{f}]$ is the same as understanding that of $\mathcal{H}^1_Z(\mathcal{O}_X)$.

The Hodge filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$ is as follows. There exists a short exact sequence

$$0 \to \mathrm{Gr}^0_V B_f \overset{t}{\to} \mathrm{Gr}^1_V B_f \overset{\tau}{\to} \mathcal{H}^1_Z(\mathcal{O}_X) \to 0$$

where the filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$ is the quotient filtration. Here the map τ is given by

$$\begin{array}{ccc}
\mathrm{Gr}^1_V B_f & \xrightarrow{\tau} & \mathcal{H}^1_Z(\mathcal{O}_X) \\
\twoheaduparrow & & \twoheaduparrow \\
V^1 B_f & & \\
\big\downarrow & & \big| \\
\mathcal{O}_X[\frac{1}{f}, s]f^s & \xrightarrow{s\mapsto -1} & \mathcal{O}_X[\frac{1}{f}]
\end{array}$$

Suppose $f = x_1^{a_1}\ldots x_n^{a_n}$ and $I_\lambda(f) = I((\lambda-\varepsilon)Z) = (x_1^{\lceil\lambda a_1\rceil-1}\ldots x_n^{\lceil\lambda a_n\rceil-1})$.

For all λ, we have $V^\lambda B_f = \sum_{j>0} D_X I_{\lambda+j}(f)\partial_t^j\delta$, and we have similar formulas for $\mathcal{F}_p \cap V^\lambda$. With $a_i = 0$ or 1, we have Z SNC. Then the Hodge filtration on $\mathcal{O}_X[\frac{1}{f}]$ is given by

$$\mathcal{F}_p\mathcal{O}_X[\frac{1}{f}] = \begin{cases} 0, & p < 0, \\ \mathcal{F}_p D_X \cdot \mathcal{O}_X[\frac{1}{f}], & p \geq 0. \end{cases}$$

Suppose Z is smooth (locally $x_1 = 0$). Then

$$\mathcal{F}_p\mathcal{O}_X(Z) = \mathcal{O}_X((p+1)Z)$$

is the pole order filtration. In general, $\mathcal{F}_p\mathcal{O}_X(Z) = \mathcal{I}_P(Z)\cdot\mathcal{O}((p+1)Z) \subseteq \mathcal{O}_X((p+1)Z)$, where $I_p(Z)$ is the p-th Hodge ideal of Z.

We can describe $\mathcal{I}_p(Z)$ using τ. Any $u \in V^1B_f$ is of form $\sum_{i=0}^p h_i\partial_t^i\delta$ as we have $\partial_t^i t^i \frac{1}{f^i}\delta = (-1)^i s(s-1)\dots(s-i+1)$ for $s = -\partial_t t$. This means $I_p(Z) = \{\sum i! h_i f^{p-i} | \sum_{i=0}^p h_i\partial_t^i\delta \in V^1B_f\}$.

As a consequence, when $p = 0$, we have $\mathcal{I}_0(Z) = \mathcal{I}((1-\varepsilon)Z)$, and $\mathcal{I}_p(Z) = \mathcal{O}_X$ if and only if for all $z \in Z$, there exists $h_p\partial_t^p\delta$+lower order terms$\in V^1B_f$ at z, if and only if $\partial_t^p\delta \in V^1B_f$, if and only if $\tilde{\alpha}(z) \geq p+1$.

Theorem 3.17. *If $Z \hookrightarrow X$ and $X \hookrightarrow X$ is smooth such that $Z|_{X'} \hookrightarrow X'$ is reduced, then*

$$I_k(Z|_{X'}) = I_k(Z)\cdot\mathcal{O}_{X'}.$$

Theorem 3.18 (Saito)**.** *If $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$ with $f+g \in \mathcal{O}(X\times Y)$, for $p \in Z(f), q \in Z(g)$, we have*

$$\tilde{\alpha}_{(p,q)}(f+g) = \tilde{\alpha}_p(f) + \tilde{\alpha}_q(g).$$

3.9. **The Du Bois complex.** Let X be a smooth projective variety. Then the complex $\Omega_X^\bullet$ and ω_X^p are fundamental objects. If X is singular, they behave badly and do not respect global geometry.

Suppose Z is reduced over $\mathbb{C}$. Du Bois and Deligne introduced $(\underline{\Omega}_Z^\bullet, F_\bullet)$ with

$$\Omega_Z^p = \mathrm{Gr}_F^p\underline{\Omega}_Z^\bullet[p] \in D^b_{\mathrm{Coh}}(Z).$$

This is the right obeject for Hodge theory as

- $\underline{\mathbb{C}}_Z \cong \underline{\Omega}_Z^\bullet$,
- The corresponding sequence $E_1^{p,q} = H^q(Z, \underline{\Omega}_Z^p \Rightarrow H^{p+q}(Z)$ degenerates at E_1 for Z proper,
- For Z projective and $\mathscr{L}$ ample, $H^q(Z, \underline{\Omega}_Z^p \otimes L) = 0$ for $p+q > \dim Z$.

There exists a canonical morphism $(\Omega_Z^\bullet, \text{stupid filtration}) \to (\underline{\Omega}_Z^\bullet, F^\bullet)$ that gives an isomorphism on Z_{sm}.

The definition of $\underline{\Omega}_Z^\bullet$ is in terms of simplicial resolutions.

3.9.1. *Steenbrink's description via log resolution.* Let $Y \to X$ be a log resolution where $\pi^{-1}(Z)$ is SNC. With an analogue of Mayer-Vietoris sequence, we have for all p and exact triangle

$$\underline{\Omega}_X^p \to \underline{\Omega}_Z^p \oplus R\pi_*\underline{\Omega}_Y^p \to R\pi_*\underline{Omega}_E^p.$$

It is known that $\underline{\Omega_E}_E^p = \Omega_Y^p/\Omega_Y^p(\log E)(-E)$. Applying the octahedral axiom gives

$$R\pi_*\underline{\Omega}_Y^p \xrightarrow{(\mathrm{Id},0)} R\pi_*\Omega_Y^p \oplus \underline{\Omega}_Z^p \xrightarrow{sum} R\pi_*\Omega_E^p$$

and exact triangle

$$R\pi_*\Omega_Y^p(\log E)(-E) \to \Omega_X^p \to \underline{\Omega}_Z^p \xrightarrow{+1}.$$

Remark 3.19. If Z has quotient or toric singularities, then

$$\underline{\Omega}_Z^p = \Omega_Z^{[p]} := j_*\Omega_{Z_{sm}}^p$$

where $j : Z_{sm} \hookrightarrow Z$. ◇

3.9.2. *Description via Hodge modules.*

Theorem 3.20 (Saito). *If $Z \hookrightarrow X_{sm}$, then*

$$\underline{\Omega}_Z^p \cong R\mathcal{H}om_{\mathcal{O}_X}(\mathrm{Gr}^{\mathcal{F}}_{p-n}\mathrm{DR}_X(i_*i^!\mathbb{Q}_X^H[n]), \omega_X)[p].$$

Sketch of proof. Set

$$\begin{array}{ccccc} U = X \setminus Z & \stackrel{j}{\hookrightarrow} & X & \stackrel{i}{\hookleftarrow} & Z \\ \uparrow \cong & & \uparrow \pi & & \uparrow \\ \pi^{-1}(U) & \stackrel{j'}{\hookrightarrow} & Y & \hookleftarrow & E \end{array}$$

This gives an exact triangle

$$\mathrm{Gr}^{\mathcal{F}}_{p-n}\mathrm{DR}_X(i_*i^!\mathbb{Q}_X^H[n]) \to \mathrm{Gr}^{\mathcal{F}}_{p-n}\mathrm{DR}_X(\mathbb{Q}_X^H[n]) \to \mathrm{Gr}^{\mathcal{F}}_{p-n}\mathrm{DR}_X\, j_*\mathbb{Q}_U^H[n] \xrightarrow{+1}.$$

Also,

$$j_*\mathbb{Q}_U^H[u] = R\pi_*j'_*\mathbb{Q}^H_{\pi^{-1}(U)}[n] = R\pi_*\Omega_Y^{n-p}(\log E)[p].$$

Apply $R\mathcal{H}om_{\mathcal{O}_X}(-,\omega_X)$, we get

$$R\mathcal{H}om_{\mathcal{O}_X}(R\pi_*\Omega_Y^{n-p}(\log E)[p], \omega_X) \to R\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{n-p}[p], \omega_X) \cong \Omega_X^p[-p] \to * \xrightarrow{+1}.$$

where the term $* \cong \Omega_Z^p[-p]$ by the exact triangle from before.

The first term is

$$R\mathcal{H}om_{\mathcal{O}_X}(R\pi_*\Omega_Y^{n-p}(\log E)[p], \omega_X) \cong R\pi_*R\mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y^{n-p}(\log E), \omega_Y)[-p] \cong R\pi_*\Omega_Y(\log E)(-E)[-p].$$

3.10. **The Du Bois complex when $\tilde{\alpha}$ is large.**

Theorem 3.21. *Let $Z \hookrightarrow X$ be a reduced hypersurface and $p \in \mathbb{Z}_{\geq 0}$. Then $\tilde{\alpha}(Z) \geq p+1$ for some $p \in \mathbb{Z}_{\geq 0}$ if and only if $\Omega_Z^i \to \underline{\Omega}_Z^i$ is an isomorphism for all $i \leq p$.*

In this case we call Z a p-Du Bois singularity. If $p = 0$, then we get (X, Z) log canonical if and only if Z has Du Bois singularity.

Lemma 3.22. *If $\tilde{\alpha}(Z) \geq p+1$, then $\mathrm{codim}_Z(Z_{sing}) \geq 2p+1$.*

Proof. If $r = \dim Z_{sing}$, we let X' be the intersection of r general hyperplanes in X and get $Z \cap X'$ is an isolated singularity. This means $\tilde{\alpha}(Z \cap X') \geq p+1$ since X' is general. Hence

$$\alpha(Z \cap X' \leq \frac{\dim X'}{2} = \frac{n-r}{2}$$

implies $n - r \geq 2p+2$. □

Proof of theorem. we know $\underline{\Omega}_Z^p \cong R\mathcal{H}om_{\mathcal{O}_X}(\mathrm{Gr}^{\mathcal{F}}_{p-n}\mathrm{DR}_X\,\mathcal{H}_Z^1(\mathcal{O}_X), \omega_X)[p+1]$. This involves graded pieces $\mathrm{Gr}_i^{\mathcal{F}}\mathcal{H}_Z^1(\mathcal{O}_X)$ for $p - n \leq i \leq p$.

Since $\tilde{\alpha}(Z) \geq p+1$, we have $\mathcal{F}_i\mathcal{H}_Z^1(\mathcal{O}_X) = \mathcal{O}_i\mathcal{H}^1Z(\mathcal{O}_X)$ for $i \leq p$, where $\mathcal{O}_i = \mathcal{O}_X((i+1)Z)/\mathcal{O}_X$. Then

$$\mathrm{Gr}_i^{\mathcal{O}}\mathcal{H}_Z^1(\mathcal{O}_X) = \mathcal{O}_Z((i+1)Z).$$

Then

$$\underline{\Omega}_Z \cong [0 \to \mathcal{O}_Z(-pZ) \to \Omega_X^1|_Z \otimes \mathcal{O}_Z(-(p-1)Z) \to \ldots \to \Omega_X^p|_Z \to 0]$$

is concentrated in degree $-p$ to 0, given by the truncated Koszul complex of $\mathcal{O}_Z \to \Omega_X|_Z \otimes \mathcal{O}_Z(Z)$ tensored by $\mathcal{O}_Z(-pZ)$. The zeroth cohomology of this compelx is Ω_Z^p, and we want other cohomologies. The Koszul complex is exact in non-zero degrees if and only if

$$\mathrm{depth}(\mathrm{Jac}_Z, \mathcal{O}_Z) \geq p \iff \mathrm{codim}_Z(Z_{sing}) \geq 2p+1.$$

So now we can apply the lemma. □

Theorem 3.23. *If $Z \hookrightarrow X_{sm}$ is a reduced hypersurface with $\tilde{\alpha}(Z) > p \geq 2$, then*

$$X \in Z \Rightarrow \mathcal{H}^{p-1}(\underline{\Omega}_Z^{n-p})_x \cong \mathcal{O}_{Z,x}/\operatorname{Jac}_Z \mathcal{O}_{Z,x}.$$

In particular, the right-hand side is non-zero if $z \in Z_{sing}$, so Z does not have quotient or toric singularity.

Theorem 3.24. *Z has p-rational singularity if and only if $\tilde{\alpha} > p+1$, so p-rational impleis p-Du Bois implies $(p-1)$-rational.*

□