DT invariants

Let S be a surface and $E \to S$ a vector bundle

The Quot scheme

$$\mathrm{Quot}_S(E, n)$$

is the moduli space of quotients

$$[E \twoheadrightarrow F] = I \leftarrow \text{kernel}$$

such that $\mathrm{rank}(F) = 0$, $c_1(F) = 0$, $\chi(F) = n$
(0-dimensional quotients of length n).

When $E = \mathcal{O}_S$,

$$\mathrm{Quot}_S(\mathcal{O}_S, n) = \mathrm{Hilb}^n(S)$$

is the Hilbert scheme of points of length n.

In this case, $\mathrm{Hilb}^n(S)$ is smooth
and the deformation obstruction theory
is $R\mathrm{Hom}(I, I)_0[1]$

$\uparrow$ trace 0

gives fundamental class $[\mathrm{Hilb}^n(S)] \in H_{2n}$.

<u>Obstruction theory</u>: a complex of v.b. $E^\bullet = [\cdots \to E^{-1} \to E^0]$
with morphism to cotangent complex $E^\bullet \xrightarrow{\varphi} L_X^\bullet$
s.t. $h^0(\varphi)$ iso, $h^{-1}(\varphi)$ surjective.

In general, $\mathrm{Quot}_S(E)$ is not smooth

the obstruction theory is

$R\mathcal{H}om(\mathcal{I}, \mathcal{F})$ (2-term for dimension reason so perfect obs thy)

gives virtual fundamental class $[\mathrm{Quot}_S(E)]^{vir} \in H_{nN}$ (Behrend-Fantechi)

Can also define virtual fundamental class for

Fano 3-folds and CY 4-folds.

but no know virtual structure for dim ≥ 5

For a vector bundle $V \to S$,

have tautological bundle

$$V^{[n]} = p_*(\mathcal{F} \otimes q^* V)$$

$$\begin{tikzcd} & \mathcal{F} \arrow[d] & & \\ & \mathrm{Quot} \times S \arrow[dl, "p"'] \arrow[dr, "q"] & & V \arrow[d] \\ \mathrm{Quot} & & & S \end{tikzcd}$$

extend to a K-theory class $\alpha \in K^0(S)$

Characteristic classes of $\alpha^{[n]}$ encodes geometric information about S.

Donaldson Thomas invariants are defined for line bundles

$$DT(L; q) = \sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}]^{vir}} e(L^{[n]})$$

More general invariants can be defined

$$C(E,V;q) = \sum q^n \int_{[\mathrm{Quot}]^{vir}} c(V^{[n]})$$

total chern class

$$S(E,V;q) = \sum q^n \int_{[\mathrm{Quot}]^{vir}} s(V^{[n]})$$

$$V(E,\alpha;q) = \sum q^n \chi^{vir}\left(\mathrm{Quot}_S(E,n), \det(V^{[n]})\right)$$

χ^{vir} denotes the virtual Euler characteristic

can be defined by $\chi(- \otimes \mathcal{O}^{vir})$ for $\mathcal{O}^{vir}$ virtual structural sheaf.

or just virtual Riemann-Roch

$$\chi^{vir}(-) = \int_{vir} td \cdot ch(-)$$

Equivariant invariants

Let S be toric surface with $T = (\mathbb{C}^*)^2$-action

For a T-rep V, consider associated bundle

$$\underset{\text{univ. bundle}}{ET \times_T V} \longrightarrow \underset{\text{classifying space}}{ET \times_T \{pt\} = BT}$$

the character classes in $H^*(BT) = H^*_T(pt)$

are equivariant character classes,

denoted $c^T, s^T, e_T, ch_T, td_T, \ldots$

Let $S=\mathbb{C}^2$. $T_0=(\mathbb{C}^*)^2=\{(t_1,t_2) \mid t_i\neq 0\}$ acts on S.

$T_1=(\mathbb{C}^*)^N=\{(y_1,\dots,y_N):y_i\neq 0\}$

Let $E=\bigoplus_{i=1}^N \mathcal{O}_S(y_i)$ rank N bundle

the point of y_i: we usually want E be a T_0-equivariant bundle over $\mathbb{C}^2$ with weights $y_1,\dots,y_N\in\mathbb{C}[t_1,t_2]$

but here we just replaced them with additional parameters from a new torus for simplicity.

$T_2=(\mathbb{C}^*)^r=\{(v_1,\dots,v_r):v_i\neq 0\}$

$V=\bigoplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$ rank r bundle.

Set $T=T_0\times T_1\times T_2$

Here $K_T(pt)=T\text{-reps}=\mathbb{Z}[\vec{t}^{\pm1},\vec{y}^{\pm1},\vec{v}^{\pm1}]$

$H_T^*(pt)=\mathbb{C}[\vec{\lambda},\vec{m},\vec{w}]$

where $\lambda=c_1^T(t)$, $m=c_1^T(y)$, $w=c_1^T(v)$

<u>Ex</u> V a T-rep of rank 3 with weight t_1, t_1y_1, t_2^2

then $c_1(V)=2\lambda_1+m_1+2\lambda_2$

$ch(V)=e^{\lambda_1}+e^{\lambda_1+m_1}+e^{2\lambda_2}$

$\updownarrow$

$V=t_1+t_1y_1+t_2^2\in K_T(\mathbb{C}^2)$.

Write $ch(\Lambda_{-1} L) = 1 - e^{c_1^T(L)} = 1 - L \in K_T(\mathbb{C}^2)$

extend by splitting principal.

Equivariant localization

If Y is complete, $\lambda \in H_T^*(Y)$, then

$$\pi_{Y*}(\lambda) = \sum_{F \text{ fixed}} \pi_{F*}\left(\frac{i_F^* \lambda}{e_T(N_F Y)}\right) \in H_T^*(pt)$$

But $Y = \mathrm{Quot}_{\mathbb{C}^2}$ is not complete. we define pushforward by the localization formula, except now it lands in $H_T^*(pt)_{loc}$

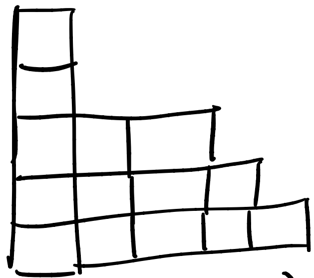
$$\int_Y : H_T^*(Y) \longrightarrow H_T^*(pt)_{loc} \cong \mathbb{C}(\vec{\lambda}, \vec{m}, \vec{w})$$

$$\alpha \longmapsto \sum_{x \in Fix} \frac{\alpha|_x}{e_T(T_x^{vir} Y)}$$

When $Y = \mathbb{C}^2$, we just have $\int_{\mathbb{C}^2} \alpha = \frac{\alpha}{\lambda_1 \lambda_2}$.

Need to know weights on the tangent of T- fixed pt of $\mathrm{Quot}_S(E)$

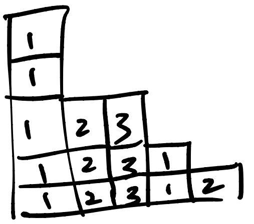
The fixed points of $\mathrm{Hilb}^n(\mathbb{C}^2)$ are given by monomial ideals $\leftrightarrow$ partitions of size n,



$(5, 3, 3, 2, 1)$, $n = 14$

Fixed pts on $\mathrm{Quot}_S(E, n)$ given by $N = \mathrm{rank}(E)$-coloured partitions of size n.

$$Z = ([Z_1], [Z_2], \ldots, [Z_N])$$



$$Z = \big((5,2), (3,1), (3) \big) \quad n = 14.$$

$\uparrow$ color 1 $\quad$ $\uparrow$ color 2 $\quad$ $\uparrow$ color 3

To compute T_Z^{vir}. Recall $T^{vir} = R\mathrm{Hom}(I, F)$

so $$T_Z^{vir} = \bigoplus_{i,j=1}^{N} \mathrm{Ext}(I_{Z_i}\langle y_i\rangle, \mathcal{O}_{Z_j}\langle y_j\rangle)$$

Let Q_i be the character of $\mathcal{O}_{Z_i}$

this computes to

$$\sum_{i,j=1}^{N} \left(Q_j - (1-t_1)(1-t_2)\overline{Q_i}\, Q_j\right) y_i^{-1} y_j$$

so we can compute $e(T_Z^{vir})$ from this

Now we want characteristic classes of $V^{[n]}$ restricted to each fixed pt Z.

$$\begin{tikzcd} V^{[n]} \arrow[d] & \mathcal{F}\otimes q_*V \arrow[d] & V \arrow[d] \\ \mathrm{Quot} & \mathrm{Quot}\times S \arrow[l] \arrow[r] & S \end{tikzcd}$$

$$\mathcal{F}|_Z = \mathcal{O}_Z = \mathcal{O}_S/I_Z$$
$$\Downarrow$$
$$V^{[n]}|_Z = \mathcal{O}_Z \otimes V.$$

$$\Rightarrow V^{[n]}|_Z = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{N} \mathcal{O}_{Z_j}(v_i y_j) = \sum_i \sum_j \sum_{(a,b)\in Z_j} v_i y_j t_1^a t_2^b.$$

$$C(E,V;q) = \sum_Z q^{|Z|} \frac{\prod_{i=1}^r \prod_{j=1}^N \prod_{(a,b)\in Z_j}(1+w_i+m_j+a\lambda_1+b\lambda_2)}{e_T(T_Z^{vir})}$$
$$\in H_T^*(pt)_{loc}.$$

$$\begin{aligned} V(E,V;q) &= \sum q^n \chi^{vir}(V^{[n]}) \\ &= \sum q^n \int_{[\mathrm{Quot}]^{vir}} td(T^{vir})\, ch(\det(V^{[n]})) \\ &= \sum_Z q^{|Z|} \frac{ch(\det(V^{[n]}|_Z))\cdot td(T_Z^{vir})}{e(T_Z^{vir})} \\ &= \sum_Z q^{|Z|} \frac{ch(\det(V^{[n]}|_Z))}{ch(\Lambda_{-1} T_Z^{vir\,\vee})} \in \mathbb{Z}[\![\vec{t}^{\pm1}, \vec{v}^{\pm1}, \vec{y}^{\pm1}]\!] \end{aligned}$$

so we say V is K-theoretic invariants.

holds for curve, surf, CY4

Segre-Verlinde Correspondence (Gottsche-Mellit for Hilb on surface)

(Bojko for Quot on curve, surf, CY4)

For Y compact, E torsion-free, $\alpha \in K^0(Y)$,

$$S_Y(E, \alpha; q) = V_Y(E, \alpha; (-1)^N q).$$

In non-compact case, S, V lie in $H_T^*(pt)_{loc} = \mathbb{C}(\vec{\lambda}, \vec{m}, \vec{w})$

we can extract and compare deg d homogeneous term.

Same correspondence holds for $d=0$,

correspondence in other degrees are expressed

as some differential equations.

Information in non-zero degrees could be used

to get reduced invariants for K3-surfaces.

(conjectured for non-compact)

Segre-Symmetry

proven for compact by Bojko

For torsion free E, V of rank N, r

$$S_Y(E, V; (-1)^N q) = S_Y(V, E; (-1)^r q)$$

Universal series structure: Ellingsrud, Göttsche, Lehn

For compact case, have series $A_1(q), A_2(q), A_3(q)$

$$S_S(E,\alpha;q) = A_1^{c_1(S)c_1(\alpha)} A_2^{c_1(S)^2} A_3^{c_1(S)c_1(E)}$$

$V = \cdots$

For non-compact, $S = \mathbb{C}^2$

$$S_S(E,\alpha;q) = \prod_{\substack{\mu,\nu,\xi \\ \text{partitions}}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha) c_\nu(S) c_\xi(E) c_1(S)}$$

$\uparrow$ degree $= |\mu| + |\nu| + |\xi| + 1$.

$$V = \prod B^{\int \cdots}$$ Proven using structures of MacDonald polynomials.

When $X = \mathbb{C}^4$ with $(\mathbb{C}^*)^4/(t_1t_2t_3t_4)$ action,

conjectured to have form

$$S = \prod A^{\int \cdots c_3(X)}$$

(line bundle case follow from Nekrasov's conjecture on DT invariants)