ETH
Eidgenössische Technische Hochschule Zürich
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Equivariant Segre and Verlinde invariants for Quot schemes

Master Thesis

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Submitted on February 19, 2023
Updated on February 22, 2023

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Abstract. Consider the toric surface $S = \mathbb{C}^2$ acted on by the torus $\mathsf{T}_0 = (\mathbb{C}^*)^2$. Let $\mathrm{Quot}_S(E,n)$ be the Quot scheme of length n quotients of the $\mathsf{T}_1 = (\mathbb{C}^*)^N$-equivariant bundle $E = \mathbb{C}^N \otimes \mathcal{O}_S$. For a $\mathsf{T} = \mathsf{T}_0 \times \mathsf{T}_1$-equivariant K-theory class $\alpha \in K_{\mathsf{T}}(S)$, we prove universal series expressions for the equivariant Segre numbers $\int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}} s(\alpha^{[n]})$ and equivariant Verlinde numbers $\chi^{\mathrm{vir}}(\mathrm{Quot}_S(E,n), \det(\alpha^{[n]}))$. Using these expressions we conclude a correspondence between the Segre and Verlinde series. When $\alpha = \mathbb{C}^r \otimes \mathcal{O}_S$, we also prove a weak symmetry in E and V. Some of the universal series shall be computed using the method of Göttsche-Mellit, which was originally developed for non-virtual invariants. When T_0 is the 1 dimensional torus $\{(t_1,t_2) : t_1t_2 = 1\}$, the toric surface $S = \mathbb{C}^2$ becomes K-trivial. We define reduced versions of the Segre and Verlinde invariants and prove a correspondence in this case. For $X = \mathbb{C}^4$ a toric Calabi-Yau 4-fold, we discuss the analogues of these invariants and generalize Cao-Kool-Monavari's cohomological limits on Hilbert schemes to Quot schemes. The Segre-Verlinde correspondence and weak symmetry are conjectured for the $X = \mathbb{C}^4$ case, as well as some vanishings of the Segre and Verlinde series in specific ranks, based on empirical data.

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1. Introduction

1.1. **Definitions of Segre and Verlinde invariants.** Let Y be a smooth quasi-projective variety. For a torsion-free sheaf E, the Quot scheme $\mathrm{Quot}_Y(E,n)$ parameterizes quotients

$$E \twoheadrightarrow F$$

such that

$$\mathrm{rank}(F)=0, c_1(F)=0, \chi(F)=n.$$

When $Y=\mathbb{C}^2$ is a toric surface or $Y=\mathbb{C}^4$ is a toric Calabi-Yau 4-fold, we shall define equivariant Segre and Verlinde invariants on these Quot schemes and find relations between them. In the non-equivariant case where Y is a smooth projective curve, surface or Calabi-Yau 4-fold, such relations are called Segre-Verlinde correspondence and are studied in [AJL$^+$21, Boj21b, Boj21a, GM22, GK22, JOP21, MOP21].

For a vector bundle V over Y, the *tautological bundle* $V^{[n]}$ on $\mathrm{Quot}_Y(E,n)$ is

$$V^{[n]} = p_*(\mathcal{F}\otimes q^*V)$$

where $p:\mathrm{Quot}_Y(E,n)\times Y\to \mathrm{Quot}_Y(E,n), q:\mathrm{Quot}_Y(E,n)\times Y\to Y$ are projections, and $\mathcal{F}$ is the universal quotient sheaf. This extends to the Grothendieck groups and associates each $\alpha\in K^0(Y)$ an $\alpha^{[n]}\in K^0(\mathrm{Quot}_Y(E,n))$. The Segre and Verlinde invariants are defined by integrating or taking Euler characteristics of various insertions of these tautological bundles.

1.1.1. *Invariants on Hilbert schemes of surfaces.* Let us begin with the case $Y=S$ a smooth projective surface and $E=\mathcal{O}_S$. The Quot scheme $\mathrm{Quot}_S(\mathcal{O}_S,n)$ is the Hilbert scheme $\mathrm{Hilb}^n(S)$ parametrizing ideal sheaves of 0-dimensional subschemes of Y of length n, which is known to be smooth projective. For $\alpha\in K^0(S)$, the *Segre and Chern series*, first defined by [Tyu94] in the study of Donaldson invariants, are respectively

$$\begin{aligned} I^{\mathcal{S}}(\alpha;q) &:= \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} s(\alpha^{[n]}),\\ I^{\mathcal{C}}(\alpha;q) &:= \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} c(\alpha^{[n]}). \end{aligned} \tag{1.1}$$

These series are related by $c(\alpha^{[n]})=s(-\alpha^{[n]})$. The *Verlinde series* is originally defined for moduli spaces of bundles on curves. Here it is defined by

$$I^{\mathcal{V}}(\alpha;q) := \sum_{n=0}^{\infty} q^n \chi(\mathrm{Hilb}^n(S), \det(\alpha^{[n]})).$$

1.1.2. *Equivariant invariants on Hilbert schemes of surfaces.* Now let S be a toric quasi-projective surface with an action of $\mathsf{T}=(\mathbb{C}^*)^2$. Details for equivariant cohomology and equivariant integration are included in Section 2.1. Here we give a quick summary. Given a T-representation V, define its *equivariant characteristic classes* by considering the associated bundle

$$E\mathsf{T}\times_{\mathsf{T}} V \to E\mathsf{T}\times_{\mathsf{T}}\{\mathrm{pt}\} = B\mathsf{T}$$

and taking its characteristic classes in $H^*(B\mathsf{T})=H^*_{\mathsf{T}}(\mathrm{pt})$. Denote $c^{\mathsf{T}}, s^{\mathsf{T}}, e_{\mathsf{T}}, \mathrm{ch}_{\mathsf{T}}, \mathrm{td}_{\mathsf{T}}$ the equivariant Chern class, Segre class, Euler class, Chern character, and Todd class respectively. From here forward, we will always use equivariant classes for toric varieties, and we will omit the torus T from the notations when it is clear from the context.

The action of T on S lifts to an action of T on $\mathrm{Hilb}^n(S)$, giving an equivariant structure to $V^{[n]}$ for any equivariant bundle V on S. For $\alpha\in K_{\mathsf{T}}(S)$, the equivariant Chern/Segre series are defined by (1.1), where the integration $\int_{\mathrm{Hilb}^n(S)}$ is replaced by *equivariant push-forward.* The result of the integration lives in the ring of fractions of the equivariant cohomology ring $H^*_{\mathsf{T}}(\mathrm{pt})$, which we denote $H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$.

For $\alpha \in K_{\mathsf{T}}(S)$, the *equivariant Verlinde series* is

$$\begin{aligned} I^{\mathcal{V}}(\alpha; q) &:= \sum_{n=0}^{\infty} q^n \chi(\mathrm{Hilb}^n(S), \det(\alpha^{[n]})) \\ &= \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} \mathrm{td}(\mathrm{Hilb}^n(S))\, \mathrm{ch}(\det(\alpha^{[n]})). \end{aligned}$$

where χ is the equivariant Euler characteristic, and the equality follows from the equivariant Riemann-Roch formula [EG99, Corollary 3.1].

1.1.3. *Virtual invariants on Quot schemes.* In general, Quot schemes of surfaces and Hilbert schemes of 3-folds and 4-folds are not smooth, in which case we do not have a fundamental class to integrate against. One way to resolve this is to work with a virtual fundamental class $[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}$.

For a smooth surface S and a torsion free sheaf E, a perfect obstruction theory for $\mathrm{Quot}_Y(E,n)$ of virtual dimension nN was constructed in [MOP15, Lemma 1]. To it, one may associate a virtual fundamental class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$ [BF98, LT96] and a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ [CFK09]. When S is compact, the virtual invariants are defined similarly as before, with the usual fundamental class $[\mathrm{Quot}_S(E,n)]$ replaced by the virtual class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$, and the Euler characteristic $\chi(\cdot)$ replaced by the virtual Euler characteristic $\chi^{\mathrm{vir}}(\cdot) := \chi(\cdot \otimes \mathcal{O}^{\mathrm{vir}})$.

Unlike for surfaces and Fano 3-folds, the usual obstruction theory for a Quot scheme of a Calabi-Yau 4-fold X is not perfect, so the previous method does not induce a virtual fundamental class. However, using a (-2)-shifted symplectic structure in the sense of [PTVV11], Borisov–Joyce [BJ15] and Oh-Thomas [OT20] constructed a virtual class $[\mathrm{Quot}_X(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})} \in H_{2nN}(\mathrm{Quot}_X(E,n), \mathbb{Z})$. Here $\mathcal{L}$ denotes the determinant line bundle $\det \mathbf{R}\mathscr{H}om_q(\mathcal{I},\mathcal{I})$ on $\mathrm{Quot}_X(E,n)$ where $\mathcal{I}$ is the universal subsheaf. As indicated by the subscript, this class is dependent on some choice of orientation $o(\mathcal{L})$, that is a choice of square root of the isomorphism

$$Q : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_{\mathrm{Quot}_X(E,n)}$$

induced by Serre duality. There is also no canonical virtual structure sheaf, and instead, we have a "twisted" structure sheaf $\hat{\mathcal{O}}^{\mathrm{vir}}$. The Verlinde numbers are defined via the "untwisted" virtual Euler characteristic $\chi^{\mathrm{vir}}(\cdot) = \chi(\cdot \otimes \det^{\frac{1}{2}}((E^\vee)^{[n]}) \otimes \hat{\mathcal{O}}^{\mathrm{vir}})$ [Boj21a, Section 1.3].

1.1.4. *Equivariant virtual invariants on Quot schemes.* To define equivariant virtual invariants on $Y = \mathbb{C}^d$, $d = 2, 4$, we first give $\mathrm{Quot}_Y(E,n)$ some torus action. Let

$$\mathsf{T}_0 = (\mathbb{C}^*)^d/(\sim) = \{(t_1, \ldots, t_d) : t_1, \ldots, t_d \neq 0\}/(\sim)$$

act on $S = \mathbb{C}^d$ naturally by scaling coordinates, where we mod out by the subgroup $\langle \sim \rangle = \langle t_1 t_2 t_3 t_4 \rangle$ when $d = 4$. Let $\mathsf{T}_1 = (\mathbb{C}^*)^N = \{(y_1, \ldots, y_n) : y_i \neq 0\}$ and $E = \oplus_{i=1}^N \mathcal{O}_Y\langle y_i \rangle$ be the T_1-equivariant bundle of rank N with weights $y_1, \ldots, y_N$. This induces a $\mathsf{T}_0 \times \mathsf{T}_1$-action on $\mathrm{Quot}_Y(E,n)$ by acting on the middle term of the sequence

$$0 \to I \to E \to F \to 0.$$

Let $\alpha \in K_{\mathsf{T}_0}(Y)$, then we can write

$$\alpha = [\oplus_{i=1}^r \mathcal{O}_Y\langle v_i \rangle] - [\oplus_{i=r+1}^{r+s} \mathcal{O}_Y\langle v_i \rangle]$$

where $v_1, \ldots, v_{r+s}$ are its T_0-weights. However, instead of thinking of v_i as T_0-weights, we would like to view them as generic parameters. Therefore we introduce an additional torus $\mathsf{T}_2 = (\mathbb{C}^*)^{r+s}$

acting on $\mathbb{C}^r \times \mathbb{C}^s$ respectively. Set $\mathsf{T} := \mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2$ and denote

$$K_{\mathsf{T}}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}; y_1^{\pm 1}, \dots, y_N^{\pm 1}; v_1^{\pm 1}, \dots, v_{r+s}^{\pm 1}]/(\sim),$$
$$H^*_{\mathsf{T}}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \dots, \lambda_d; m_1, \dots, m_N; w_1, \dots, w_{r+s}]/(\sim).$$

where we quotient by the ideals $(\sim) = (t_1t_2t_3t_4 - 1)$ and $(\sim) = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$ respectively when $d = 4$.

Recall that the non-virtual equivariant invariants are defined using equivariant localization. Similarly, the virtual equivariant invariants for $S = \mathbb{C}^2$ can be defined using virtual equivariant localization [GP97, CFK09]. For toric Calabi-Yau 4-folds, the equivariant Donaldson-Thomas invariants were first introduced in Cao-Leung [CL14, Section 8]; the K-theoretic invariants were predicted by Cao-Kool-Monavari [CKM22] and formalized by J. Oh and R. P. Thomas [OT20] using the twisted virtual structure sheaf and their virtual equivariant localization. The equivariant Segre and Verlinde numbers for $X = \mathbb{C}^4$ are special cases of these invariants. For more details of these constructions, see Sections 3.4 and 5.1.

On the Quot scheme $\mathrm{Quot}_Y(E, n)$ where $E = \oplus_{i=1}^N \mathcal{O}_Y\langle y_i\rangle$, we define *equivariant virtual Segre and Verlinde series* for $\alpha = [\oplus_{i=1}^r \mathcal{O}_Y\langle v_i\rangle] - [\oplus_{i=r+1}^{r+s} \mathcal{O}_Y\langle v_i\rangle]$ to be respectively

$$\mathcal{S}_Y(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}} s(\alpha^{[n]}),$$
$$\mathcal{C}_Y(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_Y(E,n)]^{\mathrm{vir}}} c(\alpha^{[n]}),$$
$$\mathcal{V}_Y(E, \alpha; q) := \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}}(\mathrm{Quot}_Y(E, n), \det(\alpha^{[n]})).$$

Remark 1.1. It is important to note that the above definitions, when $Y = X = \mathbb{C}^4$, are dependent on a choice of signs at each fixed point $Z \in \mathrm{Quot}_X(E, n)^{\mathsf{T}}$, which is suppressed from the notation. One could compare this to the choice of orientation in the compact case. We denote the sign at Z to be $(-1)^{o(\mathcal{L})|_Z}$, and in the equivariant setting, we call $o(\mathcal{L})$ a *choice of signs.*

When $N = 1$, the weight on $E = \mathcal{O}_Y\langle y_1\rangle$ is not necessary as this extra action does not affect the fixed locus, so we sometimes ignore it by setting $y_1 = 1$. By definition, the coefficients of the Chern and Segre series are rational functions in the cohomological parameters $\lambda_1, \dots, \lambda_d, m_1, \dots, m_N, w_1, \dots, w_{r+s}$. For the Verlinde series, they are in K-theoretic parameters $t_1, \dots, t_d, y_1 \dots, y_N, v_1 \dots, v_{r+s}$. Using the identification of Remark 2.4, they can be viewed as functions in the cohomological parameters as well. Define

$$*_{Y,i}(E, \alpha; q) := *_Y(E, \alpha; q)|_{\deg \vec{\lambda}, \vec{m}, \vec{w} = i}$$

to be the part with total degree i in those variables, for $* \in \{\mathcal{S}, \mathcal{C}, \mathcal{V}\}$. More precisely, by restricting a multi-variable function to a certain degree, we mean the following.

Definition 1.2. Let $f(z_1, \dots, z_k)$ be a function in the ring of fractions of $\mathbb{C}[\![z_1, \dots, z_k]\!]$. Consider the formal Laurent series expansion of $f(bz_1, \dots, bz_k)$ in the variable b:

$$f(bz_1, \dots, bz_k) = \sum_{i=-\infty}^{\infty} f_i(z_1, \dots z_k) b^i.$$

For $i \in \mathbb{Z}$, the part of f with *total degree* i is

$$f(z_1, \dots, z_k)|_{\deg \vec{z} = i} := f_i(z_1, \dots, z_k).$$

1.1.5. *Reduced invariants on Quot schemes of surfaces.* When S is a K-trivial surface, the obstruction on $\mathrm{Quot}_S(E,n)$ contains a trivial summand making $e(T^{\mathrm{vir}})$ vanish, as a result, the invariants all vanish. We may instead consider the reduced classes and invariants from Gromov-Witten theory and stable pair theory [KT14], which has been employed to study the enumerative geometry of Hilbert schemes in for instance [GSY17]. In the case of $\mathrm{Quot}_S(E,n)$, a reduced perfect obstruction theory can be obtained by removing one copy of $\mathcal{O}_{\mathrm{Quot}_S(E,n)}$ from the usual obstruction for Quot schemes. The equivariant analogue of a K-trivial surface would be $S=\mathbb{C}^2$ with the action of the 1-dimensional torus $\mathsf{T}_0=\{(t_1,t_2): t_1t_2=1\}$. Let T^{red} be the virtual tangent bundle obtained from the reduced obstruction theory. For $E=\oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha=[\oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle]-[\oplus_{i=r+1}^s \mathcal{O}_S\langle v_i\rangle]$, we define the *reduced Segre, Chern, Verlinde series* to be respectively

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q):=\sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} s(\alpha^{[n]}),$$

$$\mathcal{C}^{\mathrm{red}}(E,\alpha;q):=\sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} c(\alpha^{[n]}),$$

$$\mathcal{V}^{\mathrm{red}}(E,\alpha;q):=\sum_{n>0}^{\infty} q^n \int_{[\mathrm{Quot}_S(E,n)]^{\mathrm{red}}} \mathrm{td}(T^{\mathrm{red}})\,\mathrm{ch}(\det(\alpha^{[n]})).$$

1.2. **Summary of results for surfaces.**

1.2.1. *Computation of Chern series.* Consider the case $S=\mathbb{C}^2$ with the $\mathsf{T}=(\mathbb{C}^*)^2$-action. Using the tools from [GM22], we are able to compute the non-virtual equivariant Chern series for bundles of rank 2 as follows in Section 4.1.

Theorem 1.3. *For V a vector bundle of rank 2 over $S=\mathbb{C}^2$, we have*

$$I^{\mathcal{C}}(V;q)=(1+q)^{\int_S c(V)}$$

where $\int_S$ denotes equivariant push-forward to a point.

Remark 1.4. For a class $\gamma\in H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$ and an invertible power series $F(q)$, the expression $F(q)^\gamma$ is to be interpreted as

$$\exp\left(\gamma\log(F(q))\right)=\sum_{n=0}^{\infty}\frac{(\gamma\log(F(q)))^n}{n!}\in\left(\bigoplus_{i=0}^{\infty}H^i_{\mathsf{T}}(\mathrm{pt})\right)_{\mathrm{loc}}[\![q]\!].$$

In the above theorem, this gives us

$$[q^n]I^{\mathcal{C}}(V;q)=\binom{\int_S c(V)}{n}$$

In Section 4.2, we extend the computation for the above theorem to the virtual setting and obtain the Chern series of line bundles for Hilbert schemes. Note that in the non-equivariant setting, this series can be retrieved from [OP22, Corollary 15].

Corollary 1.5. *Let $S=\mathbb{C}^2$, and $L=\mathcal{O}_S\langle v_1\rangle$ a T-equivariant line bundle over S. We have*

$$\mathcal{C}_S(\mathcal{O}_S,L;q)=\left(\frac{1}{1-q}\right)^{\int_S c(L)c_1(S)}.$$

By extracting the part with the lowest total degree in λ_1,λ_2,w_1, we obtain the following 2-dimensional analogue to the Donaldson-Thomas partition function for $\mathbb{C}^3$ [MNOP06b, Theorem 1], or Cao-Kool's formulation of Nekrasov's conjecture for $\mathbb{C}^4$ [CK17, Appendix B].

Corollary 1.6 (Corollary 4.3)**.** *For $S=\mathbb{C}^2$, the following equality holds*

$$\sum_{n=1}^{\infty} q^n \int_{[\mathrm{Hilb}^n(S)]^{vir}} 1 = e^{\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}q}.$$

1.2.2. *Universal series expressions.* A common approach used to find closed formulas for the Segre and Verlinde series is by computing their universal series. In the non-virtual Hilbert scheme case for projective surfaces, it was proven in [EGL99] using methods of cobordism classes that the Segre and Verlinde invariants admit the following universal series expressions

$$I^{\mathcal{C}}(\alpha;q) = A_0(q)^{c_2(\alpha)} A_1(q)^{\chi(\det(\alpha))} A_2(q)^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3(q)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} A_4(q)^{K_S^2},$$

$$I^{\mathcal{V}}(\alpha,q) = B_1(q)^{\chi(\det(\alpha))} B_2(q)^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3(q)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} B_4(q)^{K_S^2}$$

where the products in the exponents are intersection products. The series $A_i(x), B_i(x)$ are universal in the sense that they only depend on α through its rank and are independent of the surface. Explicit formulas for these series were conjectured and computed in [Leh99, MOP17, MOP21, EGL99, GM22]. The Segre-Verlinde correspondence in this case concerns the relations between A_i and B_i. It was first proposed by D. Johnson and Marian-Oprea-Pandharipande in relation to the study of Le Potier's strange duality [MOP17, Joh18] and recently proven by Göttsche-Mellit [GM22].

For virtual invariants on Quot schemes of a smooth projective surface S and torsion free sheaf E, the universal series expressions are given by [Boj21a, Theorem 1.2]:

$$\mathcal{S}_S(E,\alpha;q) = A_1^{\mathrm{vir}}(q)^{c_1(S)c_1(\alpha)} A_2^{\mathrm{vir}}(q)^{c_1(S)^2} A_3^{\mathrm{vir}}(q)^{c_1(S)c_1(E)},$$

$$\mathcal{V}_S(E,\alpha;q) = B_1^{\mathrm{vir}}(q)^{c_1(S)c_1(\alpha)} B_2^{\mathrm{vir}}(q)^{c_1(S)^2} B_3^{\mathrm{vir}}(q)^{c_1(S)c_1(E)}.$$

The explicit formulas are computed in [AJL$^+$21, Theorem 17] for A_1, A_2, B_1, B_2 and in [Boj21a, Theorem 1.2] for A_3, B_3.

Unlike the compact case where the invariants are simply numbers, the equivariant invariants can contain terms of various degrees in $H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}$. This is reflected by the following theorem where the non-virtual equivariant Segre and Verlinde invariants are written as infinite products of series labeled by partitions. The notations for partitions are set in Section 2.2. For a partition μ and a K-theory class α, denote

$$c_\mu(\alpha) := \prod_{i=1}^{\ell(\mu)} c_{\mu_i}(\alpha).$$

Theorem 1.7 (Theorem 3.9)**.** *Let $S=\mathbb{C}^2$. For any $r\in\mathbb{Z}$, $N>0$, there exist universal power series $A_{\mu,\nu,\xi}(q), B_{\mu,\nu,\xi}(q)$, dependent on N and r, such that for $E=\oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha\in K_{\mathsf{T}}(S)$ of rank r, the equivariant virtual Segre and Verlinde series on $\mathrm{Quot}_S(E,n)$ can be written as the following infinite products*

$$\mathcal{S}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{V}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{C}_S(E,\alpha;q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)}.$$

The series in the above expressions are universal in the sense that they depend on the input α only by its rank r. Sometimes for clarity, we will add superscripts N, r to indicate the ranks of E

and α. Note that the series labeled by μ, ν, ξ are exponentiated to homogeneous rational functions in

$$H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}} = \mathbb{C}(\lambda_1, \lambda_2; m_1, \dots, m_N; w_1, \dots, w_{r+s})$$

of degree $|\mu| + |\nu| + |\xi| - 1$. The degree 0 terms occur when one of μ, ν, ξ is the partition (1) and the rest are the empty partition (0). The argument of Section 3.3 shows that the series with degree 0 exponents are necessarily equal to the series from in the projective case, that is

$$\begin{aligned} A_{(1),(0),(0)}(q) &= A_1^{\mathrm{vir}}(q), \quad A_{(0),(1),(0)}(q) = A_2^{\mathrm{vir}}(q), \quad A_{(0),(0),(1)}(q) = A_3^{\mathrm{vir}}(q), \\ B_{(1),(0),(0)}(q) &= B_1^{\mathrm{vir}}(q), \quad B_{(0),(1),(0)}(q) = B_2^{\mathrm{vir}}(q), \quad B_{(0),(0),(1)}(q) = B_3^{\mathrm{vir}}(q). \end{aligned} \tag{1.2}$$

The universal series expressions of the reduced invariants take a much simpler form; as opposed to having series exponentiated to some powers of cohomology classes, we have the following additive expressions.

Theorem 1.8 (Theorem 3.15)**.** *When $S = \mathbb{C}^2$, the equivariant reduced Segre and Verlinde series for $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha \in K_{\mathsf{T}}(S)$ are*

$$\begin{aligned} \mathcal{S}^{red}(E, \alpha; q) &= \sum_{\mu,\nu,\xi} \log\left(A_{\mu,\nu,\xi}(q)\right) \cdot \int_S c_\mu(\alpha) c_\nu(S) c_\xi(E), \\ \mathcal{V}^{red}(E, \alpha; q) &= \sum_{\mu,\nu,\xi} \log\left(B_{\mu,\nu,\xi}(q)\right) \cdot \int_S c_\mu(\alpha) c_\nu(S) c_\xi(E), \\ \mathcal{C}^{red}(E, \alpha; q) &= \sum_{\mu,\nu,\xi} \log\left(C_{\mu,\nu,\xi}(q)\right) \cdot \int_S c_\mu(\alpha) c_\nu(S) c_\xi(E) \end{aligned}$$

where $A_{\mu,\nu,\xi}, B_{\mu,\nu,\xi}$ and $C_{\mu,\nu,\xi}$ are the same series from Theorem 1.7.

1.2.3. *Virtual Segre-Verlinde correspondence.* When Y is compact, the virtual Segre-Verlinde correspondence has been proven for compact surfaces and Calabi-Yau 4-folds [Boj21a, Theorem 1.6] for torsion free sheaves E to be

$$\mathcal{S}_Y(E, \alpha; q) = \mathcal{V}_Y(E, \alpha; (-1)^N q).$$

As a corollary to Theorem 1.7 and the relations (1.2), we prove the following "weak" equivariant Segre-Verlinde correspondence.

Corollary 1.9 (Corollary 3.11)**.** *In the setting of Theorem 1.7, we have the following correspondence*

$$A_{\mu,\nu,\xi}(q) = B_{\mu,\nu,\xi}((-1)^N q).$$

whenever one of μ, ν, ξ is (1) and the other two are (0). In particular, the degree 0 part satisfies

$$\mathcal{S}_{S,0}(E, \alpha; q) - \mathcal{V}_{S,0}(E, \alpha; (-1)^N q) = \sum_{n=2}^{\infty} \frac{f_n}{(\lambda_1\lambda_2)^{n-2}} \cdot \left(\int_S c_1(S)\right)^2 \cdot q^n$$

for some terms $f_n \in H^{2n-2}_{\mathsf{T}}(pt)$ dependent on α.

This is weak in the sense that only the series whose powers are degree 0 satisfy the usual correspondence. Computation for small values of n shows that the other series might not agree, i.e. the terms f_n can be non-zero. Hence the "strong" Segre-Verlinde correspondence does not hold for $\mathbb{C}^2$ in the equivariant setting. One might ask whether there is any relations between the series whose powers have non-zero degrees. To answer this, we compute some identities satisfied by

$C_{\mu,(0),\xi}, B_{\mu,(0),\xi}$ in Section 4.3. Before stating these identities, some notations are needed. Given any partition μ, integer $a \in \mathbb{Z}$ and $n > 0$, the binomial coefficients for μ is

$$\binom{a}{\mu} := \prod_{i=1}^{\ell(\mu)} \binom{a}{\mu_i},$$

and the downward factorial of a by n is

$$(a)_{(n)} := a \cdot (a-1) \cdots (a-n+1).$$

Furthermore, set

$$(a)_{(0)} = 1, \quad (a)_{(-1)} = (a+1)^{-1}.$$

For multiple integers $n_1, n_2, \ldots, n_m$, the multinomial coefficient is

$$\binom{a}{n_1, \ldots, n_m} = \frac{a!}{n_1! \ldots n_m! \cdot (a - \sum_i n_i)!}.$$

Theorem 1.10 (Theorem 4.5)**.** *For rank $r \neq 0$ and $n, k > 0$, the universal series of Theorem 3.9 satisfy the following identities*

$$[q^m] \sum_{|\mu|=k} \binom{|r|}{\mu} \log C_{\mu,(0),(0)}(q,z) = \frac{|r|}{mk} \binom{mr-1}{mN-1} \binom{m(r-N)}{k-1},$$

$$[q^m] \log B_{(1)_k,(0),(0)}(q) = \frac{m^{k-2}}{k!} \binom{m(r+N)-1}{mr},$$

$$[q^m] \sum_{|\xi|=k} \binom{N}{\xi} \log C_{(0),(0),\xi}(q) = \frac{r}{mk} \binom{mr-1}{mN-1} \binom{m(r-N)}{k-1},$$

$$[q^m] \sum_{|\xi|=k} \binom{N}{\xi} \log B_{(0),(0),\xi}(q) = \frac{|r|^{k-1} N m^{k-2}}{k!} \binom{m(r+N)-1}{mN}$$

where $(1)_k = (1, \ldots, 1)$ is the partition with k copies of 1*.*

When $r > 0$, we have

$$[q^m] \sum_{|\mu|=k_1} \sum_{\xi=k_2} \binom{r}{\mu} \binom{N}{\xi} \log C_{\mu,(0),\xi}(q) = \frac{r(r-N)}{k_1 k_2} \binom{mr-1}{mN-1} \binom{m(r-N)-1}{k_1-1, k_2-1},$$

$$[q^m] \sum_{\xi=k_2} \binom{N}{\xi} \log B_{(1)_{k_1},(0),\xi}(q) = \frac{r^{k_2} m^{k_1+k_2-2}}{k_1! k_2!} \binom{m(r+N)-1}{mr}.$$

In particular, setting $k = 1$, the first two equalities of this theorem give

$$C^{-r,N}_{(1),(0),(0)}(q) = -B^{r,N}_{(1),(0),(0)}((-1)^N q).$$

This is consistent with the Segre-Verlinde correspondence in degree 0 from Corollary 1.9 because by the fact that $\mathcal{S}(E, \alpha; q) = \mathcal{C}(E, -\alpha; q)$, we have

$$A^{r,N}_{(1),(0),(0)}(q) = \left(C^{-r,N}_{(1),(0),(0)}(q)\right)^{-1}.$$

Furthermore, when $k = 2$, we have the following correspondence in degree 1.

Corollary 1.11 (Corollary 4.6)**.** *The universal series of Theorem 3.9 satisfy the following correspondence*

$$\left(C^{-r,N}_{(1,1),(0),(0)}(q)\right)^{r^2} \left(C^{-r,N}_{(2),(0),(0)}(q)\right)^{\binom{|r|}{2}} = \left(B^{r,N}_{(1,1),(0),(0)}((-1)^N q)\right)^{|r|(r+N)}.$$

$$\left(C^{-r,N}_{(0),(0),(1,1)}(q)^{r^2} C^{-r,N}_{(0),(0),(2)}(q)^{\binom{|r|}{2}}\right)^{|r|} = \left(B^{r,N}_{(0),(0),(1,1)}(-q)^{r^2} B^{r,N}_{(0),(0),(2)}(-q)^{\binom{|r|}{2}}\right)^{r+N}.$$

Also proven in [Boj21a, Theorem 1.7] is a symmetry for virtual Segre series, which states

$$\mathcal{S}_Y(E,V;(-1)^N q) = \mathcal{S}_Y(V,E;(-1)^r q)$$

for torsion free sheaves E and V of rank N and r respectively. Similar to the Corollary 1.9, we have the following weak version of this symmetry.

Corollary 1.12. *In the setting of Theorem 1.7, for $\alpha = V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$, we have the following symmetry*

$$A^{N,r}_{\mu,\nu,\xi}((-1)^N q) = A^{r,N}_{\xi,\nu,\mu}((-1)^r q)$$

whenever one of μ,ν,ξ is (1) and the other two are (0). In degree 0, we have

$$\mathcal{S}_{S,0}(E,V;(-1)^N q) - \mathcal{S}_{S,0}(V,E;(-1)^r q) = \sum_{n=1}^{\infty} \frac{g_n}{(\lambda_1\lambda_2)^{n-2}} \cdot \left(\int_S c_1(S)\right)^2 \cdot q^n$$

for some terms $g_n \in H^{2n-2}_{\mathsf{T}}(pt)$.

A computer calculation for cases

$$\begin{cases} n=1, \text{for } N\le 5,\ r\le 3, \\ n=2, \text{for } N\le 3,\ r\le 3, \\ n=3, \text{for } N\le 3,\ r\le 2, \\ n=4,5, \text{for } N\le 2,\ r=1 \end{cases}$$

suggests that the strong version of the symmetry holds for equivariant Segre series, i.e.

$$\mathcal{S}_{\mathbb{C}^2}(E,V;(-1)^N q) = \mathcal{S}_{\mathbb{C}^2}(V,E;(-1)^r q).$$

In Theorem 1.10, if we swap N and r for the identity involving $C_{(0),(0),\xi}$, we would get the exact identity for $C_{\mu,(0),(0)}$. This also suggests that the strong Segre symmetry holds.

As for the Verlinde series, in the (non)-equivariant case, the (weak) Segre symmetry together with the (weak) Segre-Verlinde correspondence would imply a (weak) Verlinde symmetry. However, the "strong" Verlinde symmetry does not hold for $S = \mathbb{C}^2$, which can be observed from the asymmetry of the series in Theorem 1.10.

1.2.4. *Reduced Segre-Verlinde correspondence.* For the reduced invariants on $S = \mathbb{C}^2$, let us denote $\mathcal{S}_i^{\mathrm{red}}$ and $\mathcal{V}_i^{\mathrm{red}}$ the degree i parts of the reduced Segre and Verlinde series respectively, in the sense of Definition 1.2. In this setting, the fact that the series from Theorem 1.8 are the same ones from Theorem 1.7 gives us the following reduced Segre-Verlinde correspondence and Segre symmetry in degree -1.

Corollary 1.13. *In the setting of Theorem 1.8, we have the following correspondence*

$$\mathcal{S}^{red}_{-1}(E,V;q) = \mathcal{V}^{red}_{-1}(E,\alpha;(-1)^N q).$$

When $\alpha = V$ is an equivariant vector bundle, we have the following symmetry

$$\mathcal{S}^{red}_{-1}(E,V;(-1)^N q) = \mathcal{S}^{red}_{-1}(V,E;(-1)^r q).$$

Let $\alpha \in K_{\mathsf{T}}(S)$ with rank r. Write $c_1\lambda = c_1(\alpha)$ and $c_2\lambda^2 = c_2(\alpha)$ for some $c_1, c_2 \in \mathbb{Q}$, then we can use the Theorem 1.8 to show that for some series $A_2(q), A_1(q), A_0(q), B_1(q), B_0(q)$, dependent on r and N,

$$\begin{aligned}
\mathcal{S}_0^{\mathrm{red}}(E,\alpha;q)|_{m_1=\cdots=m_N=0} &= -\log\left(A_{(2),(0)}(q)\right)\cdot c_2 - \log\left(A_{(1,1),(0)}(q)\right)\cdot c_1^2 + \log\left(A_{(0),(2)}(q)\right) \\
&=: A_2(q)c_2 + A_1(q)c_1^2 + A_0(q), \\
\mathcal{V}_0^{\mathrm{red}}(E,\alpha;q)|_{m_1=\cdots=m_N=0} &= -\log\left(B_{(1,1),(0)}(q)\right)\cdot c_1^2 + \log\left(B_{(0),(2)}(q)\right) \\
&=: B_1(q)c_1^2 + B_0(q).
\end{aligned}$$

Using the identities of Theorem 1.8, we can get the corresponding relations in the reduced case. For example, Corollary 1.11 gives a Segre-Verlinde correspondence in degree 0. Furthermore, we have the following formula for $B_1(q)$ from Theorem 1.10 by setting $k = 2$.

Corollary 1.14. *The series $B_1(q)$ is explicitly given by*

$$[q^n]B_1(q) = -\frac{1}{2}\binom{(N+r)n-1}{rn}$$

for $n > 0$ and $r \neq 0$. The binomial coefficient $\binom{n}{k}$ for negative input k is $[x^{-k}](1+x^{-1})^n$.

We give some conjectural formulas for A_0, B_0 in terms of A_1, B_1 when $N = 1$, respectively, checked for $n \leq 20, r < 5$.

Conjecture 1.15. *When $N = 1$ and $r \in \mathbb{Z}$, we have*

$$[q^n]B_0(q) = \frac{\binom{r+1}{2}(n-1)-1}{6}[q^n]B_1(q).$$

When $N = 1$, $r < -1$ and $n > 1$, we have

$$[q^n]A_0(q) = \frac{1}{12}r(nr+n+2)[q^n]A_1(q).$$

1.3. **Correspondence for 4-folds and other observations.**

1.3.1. *Segre-Verlinde correspondence.* Since all toric Calabi-Yau 4-folds are non-compact, we do not know how the invariants in the non-compact case relate to the ones in the compact case. We have seen for the surface case the powers on the universal series have a factor of $c_1(S)$. Considering the series given in [Boj21b, Proposition 4.13] and [Boj21a, Equation (3.38)], together with Section 5.3, we see that this term should be replaced by $c_3(X)$ in the 4-fold case. Motivated by Corollary 1.9 and 1.12, we conjecture a weak Segre-Verlinde correspondence and symmetry for $X = \mathbb{C}^4$. The correspondence part was checked with a computer program for

$$\begin{cases} n \leq 6, & \text{for } N, r \leq 1, \\ n \leq 3, & \text{for } N, r \leq 2, \\ n \leq 2, & \text{for } N, r \leq 3, \\ n \leq 2, & \text{for } N, r \leq 4. \end{cases}$$

The symmetry part was checked for

$$\begin{cases} n \leq 4, & \text{for } N = r = 1, \\ n \leq 3, & \text{for } N = 1, r = 2, \\ n \leq 2, & \text{for } N = r = 2, \\ n \leq 2, & \text{for } N = 1, r = 3. \end{cases}$$

Conjecture 1.16. *Let $X = \mathbb{C}^4$, $E = \oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$, $V = \oplus_{i=1}^r \mathcal{O}_X\langle v_i\rangle$, and $\alpha \in K_{\mathsf{T}}(X)$, then for some choice of signs $o(\mathcal{L})$, we have the following symmetry and correspondence*

$$\mathcal{S}_X(E, V; (-1)^N q) = \mathcal{S}_X(V, E; (-1)^r q),$$

$$\mathcal{S}_{X,0}(E, \alpha; q) - \mathcal{V}_{X,0}(E, \alpha; (-1)^N q) = \sum_{n=1}^{\infty} \frac{F_n}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{n-2}} \cdot \left(\int_X c_3(X)\right)^2 \cdot q^n$$

for some terms $F_n \in H^{4n-6}_{\mathsf{T}}(pt)$ dependent on α.

1.3.2. *Nekrasov's conjectures.* In [CK17, Appendix B], Y. Cao and M. Kool gave the following formulation of Nekrasov's conjecture [Nek20, Section 5]. We generalize this to Quot schemes of $\mathbb{C}^2$ and $\mathbb{C}^4$ as follows.

Conjecture 1.17. *Let $Y = \mathbb{C}^d$, $E = \oplus_{i=1}^{r+1} \mathcal{O}_Y\langle y_i\rangle$, and $V = \oplus_{i=1}^r \mathcal{O}_Y\langle v_i\rangle$. When $d = 2$, or $d = 4$ with some choice of signs $o(\mathcal{L})$, we have*

$$\mathcal{C}_Y(E, V; q) = \exp\left(q \int_Y c_{d-1}(Y)\right).$$

Note that when $N = 1$ and $Y = \mathbb{C}^2$, this is exactly Corollary 1.6. When $Y = \mathbb{C}^4$, we shall show that this conjecture is a consequence of Nekrasov-Piazzalunga's conjecture [NP19, Section 2.5] using a Quot scheme version of Cao-Kool-Monavari's cohomological limit [CKM22, Appendix A] in Proposition 5.9. For $Y = \mathbb{C}^2$, we check it for $N = 2, 3, 4, 5$ up to and including $n = 7, 4, 2, 2$ respectively.

Since $[\mathrm{Quot}_Y(E, n)]^{\mathrm{vir}}$ has virtual dimension nN, we have $C_Y^N(V; q) = 1$ when $N > r$ in the compact case simply for degree reason, which means the corresponding Verlinde series is also trivial due to the Segre-Verlinde correspondence. In the non-compact case, the above conjectures suggest that they may contain negative degree terms. However, with a computer calculation for $N = 2, 3, 4, 5$ and all possible r, up to and including $n = 8, 5, 2, 2$, we see a complete vanishing when $N > r + 1$ for Chern numbers, and when $r < N$ for rank $-r$ Verlinde numbers.

Conjecture 1.18. *Let $Y = \mathbb{C}^d$, $N > 1$, and $E = \oplus_{i=1}^N \mathcal{O}_Y\langle y_i\rangle$. When $d = 2$, or $d = 4$ with some choice of signs, we have for $r = 0, 1, \ldots, N-2$ and $V = \oplus_{i=1}^r \mathcal{O}_Y\langle v_i\rangle$,*

$$\mathcal{C}_Y(E, V; q) = 1.$$

Furthermore, for $r = 1, \ldots, N-1$, we have

$$\mathcal{V}_Y(E, -[V]; q) = 1.$$

In Proposition 5.9, we also show that the Chern series part of this conjecture for $d = 4$ is a consequence of Nekrasov-Piazzalunga's Conjecture 5.7.

2. Preliminaries

2.1. **Equivariant cohomology and K-theory.** Given a topological group G acting on a topological space M, the *equivariant cohomology* $H^*_G(M)$ is defined to be $H^*(EG \times M/G)$, where $EG \to BG$ is the universal principle G-bundle on the classifying space BG. The map $M \to \mathrm{pt}$ induces a ring homomorphism $H^*_G(\mathrm{pt}) \to H^*_G(M)$, making $H^*_G(M)$ a module over $H^*_G(\mathrm{pt})$ for any M, and we can view $H^*_G(\mathrm{pt})$ as a "coefficient ring".

Definition 2.1. Given a G representation V, viewed as a vector bundle $V \to \{\mathrm{pt}\}$, we define its *equivariant characteristic classes* by taking the associated bundle

$$EG \times_G V \to EG \times_G \{\mathrm{pt}\} = BG$$

and taking its characteristic classes in $H^*(BG) = H^*_G(\mathrm{pt})$. Denote $c_i^G, e_G, \mathrm{ch}_G, \mathrm{td}_G$ the equivariant versions of the i-th Chern class, the Euler class, the Chern character, and the Todd class respectively.

Example 2.2. For the action of a d-dimensional torus $\mathsf{T} = (\mathbb{C}^*)^d = \{(t_1, \dots, t_d) : t_i \neq 0\}$, the coefficient ring is

$$H^*_{\mathsf{T}}(\mathrm{pt}) = H^*(B\mathsf{T}) = H^*((\mathbb{C}P^\infty)^d) = \mathbb{C}[\lambda_1, \dots, \lambda_d]$$

and $\lambda_1, \dots, \lambda_d$ are exactly the equivariant first Chern classes of 1-dimensional T-representations with weight $t_1, \dots, t_d$ respectively. In general [Edi97, Section 3.2],

$$c_1^{\mathsf{T}}(\mathbb{C}\langle t_1^{w_1} \dots t_d^{w_d}\rangle) = w_1\lambda_1 + \cdots + w_d\lambda_d.$$

For the $d-1$-dimensional subtorus $\mathsf{T}' = \{(t_1, \dots, t_d) \in \mathbb{C}^d : t_1 \dots t_d = 1\} \subseteq \mathsf{T}$, the inclusion induces the following isomorphism to the quotient ring:

$$H^*_{\mathsf{T}'}(\mathrm{pt}) \cong \mathbb{C}[\lambda_1, \dots, \lambda_d]/(\lambda_1 + \cdots + \lambda_d).$$

We can construct the *equivariant K-group* $K_G(M)$ from the G-equivariant vector bundles. When $M = \mathbb{C}^d$, vector bundles over M are trivial, but they may carry non-trivial G-actions. Therefore the equivariant bundles on $\mathbb{C}^d$ correspond to finite-dimensional G-representations. The equivariant characteristic classes of vector bundles can then be extended to the K-theory classes. For example, the Euler class of $\alpha = [V] - [W] \in K_G(\mathrm{pt})$ is $e^G(\alpha) = e^G(V)/e^G(W)$, which lives in the ring of fractions $H^*_G(\mathrm{pt})_{\mathrm{loc}}$; the Chern character is $\mathrm{ch}_G(\alpha) = \mathrm{ch}_G(V) - \mathrm{ch}_G(W)$, which lives in $\prod_{i=0}^\infty H^i_G(\mathrm{pt})$.

Example 2.3. When $Y = \mathbb{C}^d$ with the natural action by $\mathsf{T} = (\mathbb{C}^*)^d$ or $\mathsf{T}' = (\mathbb{C}^*)^{d-1}$, we have the following character rings:

$$K_{\mathsf{T}}(Y) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] \cong K_{\mathsf{T}}(\mathrm{pt}),$$

$$K_{\mathsf{T}'}(Y) \cong \frac{\mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]}{(t_1 \cdots t_d - 1)} \cong K_{\mathsf{T}'}(\mathrm{pt})$$

where for any weight $w = (w_1, \dots, w_d)$, the line bundle $\mathcal{O}_Y\langle t^w\rangle := \mathcal{O}_Y \otimes t^w$ simply corresponds to its character $t^w = t_1^{w_1} \cdots t_d^{w_d}$.

Remark 2.4. We will occasionally identify Chern characters, which are power series in cohomology, with elements in K-theory by

$$t_1^{w_1} \cdots t_d^{w_d} \leftrightarrow \mathrm{ch}_{\mathsf{T}}(\mathcal{O}_Y\langle t_1^{w_1} \cdots t_d^{w_d}\rangle) = e^{w_1\lambda_1 + \cdots + w_d\lambda_d}.$$

This allows us to consider certain classes in cohomology as elements of $K_{\mathsf{T}}(\mathrm{pt})$. For example, for the line bundle $L = \mathcal{O}_Y\langle t_1^{w_1} \cdots t_d^{w_d}\rangle$, we write

$$\mathrm{ch}_{\mathsf{T}}(\Lambda_{-1}L^\vee) = 1 - e^{-c_1^{\mathsf{T}}(L)} = 1 - t_1^{-w_1} \cdots t_d^{-w_d} \in K_{\mathsf{T}}(Y).$$

The reason we consider equivariant cohomology is for equivariant integration. The integration formula of [EG95b, Corollary 1] via equivariant localization states that on a smooth complete variety Y with the action of a torus T, for λ an equivariant cohomological class, we have

$$\pi_{Y*}(\lambda) = \sum_F \pi_{F*}\left(\frac{i_F^*\lambda}{e_{\mathsf{T}}(N_F Y)}\right)$$

where the sum goes through the components F of the fixed locus, $N_F Y$ denotes the normal bundle, π denotes projection to a point, and i denotes the inclusion map. The right hand side of this formula can be used to define equivariant integration in general; for Y an arbitrary smooth variety with finitely many fixed (reduced) points, the *equivariant push-forward* of π_Y is

$$\begin{aligned} \int_Y : H^*_{\mathsf{T}}(Y) &\to H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}, \\ \alpha \quad &\mapsto \sum_{x \in Y^{\mathsf{T}}} \frac{i_x^* \alpha}{e_{\mathsf{T}}(T_x Y)}. \end{aligned} \tag{2.1}$$

Example 2.5. Again let $Y = \mathbb{C}^d$ with the natural $\mathsf{T} = (\mathbb{C}^*)^d$-action. The only T-fixed point of Y is the origin. At the origin, the character for the tangent space is $T_0 Y = t_1 + t_2 + \cdots + t_d \in K_T(\mathrm{pt})$, so $e_{\mathsf{T}}(T_0 Y) = \lambda_1 \cdots \lambda_d$. Substituting into (2.1), we have

$$\int_Y \alpha = \frac{\alpha}{\lambda_1 \cdots \lambda_d}$$

2.2. **Partitions and solid partitions.** A *partition* μ is a finite sequence $(\mu_1, \mu_2, \ldots, \mu_\ell)$ of non-increasing positive integers. The size $|\mu|$ is the sum of μ_i's and we call $\ell = \ell(\mu)$ its length. The empty sequence (0) is the *empty partition* with size $|(0)| = 0$. Each partition μ corresponds to a young diagram which consists of pairs of non-negative integers $(i, j) \in \mathbb{Z}^2_{\geq 0}$ as follows

$$\mu \longleftrightarrow \{(i, j) : j < \mu_{i+1}\}.$$

A pair $\square = (i, j)$ in the above set is called a box in μ, which we denote $\square \in \mu$. The *conjugate partition* μ^t is defined to be the partition whose boxes are $\{(j, i) : (i, j) \in \mu\}$. Denote $c(\square), r(\square), a(\square), l(\square)$ the *column index*, *row index*, *arm length* and *leg length* of $\square = (i, j) \in \mu$, defined explicitly as follows

$$\begin{aligned} &c(\square) = j, \quad r(\square) = i, \\ &a(\square) = \mu_{i+1} - j - 1, \quad l(\square) = \mu^t_{j+1} - i - 1. \end{aligned}$$

When $i, j > 0$, a necessary condition for box (i, j) to be in μ is that both $(i-1, j)$ and $(i, j-1)$ are in μ. When $i = 0$ (resp. $j = 0$), we only need $(i, j-1) \in \mu$ (resp. $(i-1, j) \in \mu$).

A *solid partition* π is a finite sequence $(\pi_{ijk})_{i,j,k \geq 1}$ of positive integers such that

$$\pi_{ijk} \geq \pi_{i+1,j,k}, \quad \pi_{ijk} \geq \pi_{i,j+1,k}, \quad \pi_{ijk} \geq \pi_{i,j,k+1}.$$

The size of $|\pi|$ is the sum of the π_{ijk}'s. As a 4-dimensional analogue to partitions, the solid partition can also be viewed as a collection of boxes

$$\pi \longleftrightarrow \{(i, j, k, l) : l < \pi_{i,j,k}\} \subseteq \mathbb{Z}^4_{\geq 0}.$$

Similar to partitions, we have

$$(i, j, k, l) \in \pi \text{ implies } \begin{cases} (i-1, j, k, l) \in \pi \text{ unless } i = 0, \\ (i, j-1, k, l) \in \pi \text{ unless } j = 0, \\ (i, j, k-1, l) \in \pi \text{ unless } k = 0, \\ (i, j, k, l-1) \in \pi \text{ unless } l = 0. \end{cases} \tag{2.2}$$

For a positive integer N, an *N-colored partition* of size n is an N-tuple of partitions $\mu = (\mu^{(1)}, \ldots, \mu^{(N)})$ such that $|\mu| := \sum |\mu^{(i)}| = n$. Figure 2.1 illustrates how the partitions are coloured based on their index. Similarly, an *N-colored solid partition* is an N-tuple of solid partitions.

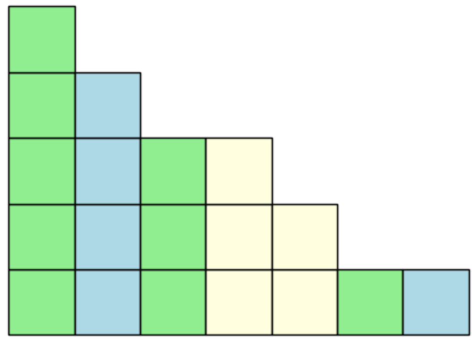

FIGURE 2.1. A 3-colored partition $\mu = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ of size $|\mu| = 19$ where $\mu^{(1)} = (5,3,1)$, $\mu^{(2)} = (4,1)$, $\mu^{(3)} = (3,2)$ are colored by green, blue and yellow respectively

2.3. **Admissible functions and universal series.** We consider the notion of admissibility in the sense of [Mel18], which will be an important condition in finding universal series for equivariant invariants.

Definition 2.6. Let $F(Q_1, Q_2 \dots; q_1, \dots, q_d) \in \mathbb{Q}(q_1, \dots, q_d)[\![Q_1, Q_2, \dots]\!]$ be a series in finitely many variables $Q_1, Q_2, \dots$ with constant term equal to 1. Then using the plethystic exponential Exp, we can write

$$F = \operatorname{Exp}\left(\frac{L}{(1-q_1)\cdots(1-q_d)}\right)$$

such that L is a power series in the variables $Q_1, Q_2, \dots$ whose coefficients are rational functions in $q_1, \dots, q_d$. The series F is called *admissible with respect to the variables* $q_1, \dots, q_d$ if the coefficients of L are polynomials in $q_1, \dots, q_d$.

Suppose $F(Q; m_1, \dots, m_N; w_1, \dots, w_r; q_1, \dots, q_d) \in \mathbb{Q}(q_1, \dots, q_d)[\![Q; m_1, \dots, m_N; w_1, \dots, w_r]\!]$ is admissible with respect to $q_1, \dots, q_d$ with constant term 1, we have the following Laurent expansion

$$\log F(Q; \vec{m}; \vec{w};; e^{\lambda_1}, \dots, e^{\lambda_d}) = \sum_{k_1, \dots, k_d = -\infty}^{\infty} H_{k_1, \dots, k_d}(Q; \vec{m}; \vec{w}) \lambda_1^{k_1} \dots \lambda_d^{k_d}.$$

Since F is admissible, by the definition of plethystic exponential,

$$(1-q_1)\cdots(1-q_d) \log F(Q; \vec{m}; \vec{w}; \vec{q})$$

is regular in a neighbourhood of $q_1 = \cdots = q_d = 0$ as a power series in $q_1, \dots, q_d$, meaning we have a lower bound $k_1, \dots, k_d \geq -1$ for the above summation.

Furthermore, suppose F is symmetric in $w_1, \dots, w_r$ and symmetric in $m_1, \dots, m_N$, then we can expand in the following elementary symmetric polynomial basis:

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \dots, e^{\lambda_d}) = \sum_{\substack{\mu, \xi \text{ partitions} \\ k_1, \dots, k_d \geq -1}} H_{\mu, \xi, \vec{k}}(Q) \prod_{i=1}^{\ell(\mu)} e_{\mu_i}(\vec{w}) \prod_{i=1}^{\ell(\xi)} e_{\xi_i}(\vec{m}) \lambda_1^{k_1} \dots \lambda_d^{k_d}$$

for some series $H_{\mu, \xi, \vec{k}}$.

Let $Y = \mathbb{C}^d$ and

$$\mathsf{T}_0 = (\mathbb{C}^*)^d, \mathsf{T}_1 = (\mathbb{C}^*)^N, \mathsf{T}_2 = (\mathbb{C}^*)^r$$

with the natural actions on $Y, E = \mathbb{C}^N \otimes \mathcal{O}_Y, V = \mathbb{C}^r \otimes \mathcal{O}_Y$ respectively. Denote $\mathsf{T} = \mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2$. Say the equivariant cohomology ring of T is $H^*_{\mathsf{T}}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \dots, \lambda_d; m_1, \dots, m_N; w_1, \dots, w_r]$. Then

V as a T-equivariant bundle has equivariant Chern roots $w_1, \ldots, w_r$, and E has Chern roots $m_1, \ldots, m_N$, so $e_i(m_1, \ldots, m_N) = c_i^{\mathsf{T}}(E), e_i(w_1, \ldots, w_r) = c_i^{\mathsf{T}}(V)$. Therefore

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\substack{\mu,\xi \text{ partitions} \\ k_1,\ldots,k_d \geq -1}} H_{\mu,\xi,\vec{k}}(Q) c_\mu(V) c_\xi(E) \lambda_1^{k_1} \ldots \lambda_d^{k_d}.$$

For $\vec{k} = (k_1, \ldots, k_d)$ where $k_1, \ldots, k_d \geq -1$, there exist polynomials $E_{\vec{k}}$ such that

$$\frac{1}{d!} \sum_{\tau \text{ permutation}} \lambda_1^{k_{\tau(1)}} \cdots \lambda_d^{k_{\tau(d)}} = \frac{E_{\vec{k}}(e_1(\lambda_1, \ldots, \lambda_d), \ldots, e_d(\lambda_1, \ldots, \lambda_d))}{\lambda_1 \cdots \lambda_d}$$

Now suppose F is symmetric in the variables $q_1, \ldots, q_d$, so $H_{\mu,k} = H_{\mu,\tau(k)}$ for any permutation τ. Hence

$$\begin{aligned} &\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) \\ &= \sum_{\substack{\mu \text{ partition} \\ k_1,\ldots,k_d \geq -1}} H_{\mu,\xi,\vec{k}}(Q) E_{\vec{k}}(e_1(\lambda_1, \ldots, \lambda_d), \ldots, e_d(\lambda_1, \ldots, \lambda_d)) c_\mu(V) c_\xi(E) \end{aligned}$$

Note the equivariant weights of the tangent space $T_0 Y$ are exactly $\lambda_1, \ldots, \lambda_d$, so $e_i(\lambda_1, \ldots, \lambda_d) = c_i^{\mathsf{T}}(Y)$. By Example 2.5, we have

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\substack{\mu \text{ partition} \\ k_1,\ldots,k_d \geq -1}} H_{\mu,\xi,\vec{k}}(Q) \int_Y E_{\vec{k}}(c_1(Y), \ldots, c_d(Y)) c_\mu(V) c_\xi(E) \tag{2.3}$$

Redistributing the terms, we get

$$\log F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_d}) = \sum_{\mu,\nu,\xi \text{ partitions}} H_{\mu,\nu,\xi}(Q) \int_Y c_\nu(Y) c_\mu(V) c_\xi(E)$$

for some series $H_{\mu,\nu,\xi}$. Exponentiate both sides, and we obtain the following *universal series expression* for F.

Proposition 2.7. *Let $F(Q; \vec{m}; \vec{w}; \vec{q}) \in \mathbb{Q}(q_1, \ldots, q_d)[\![Q; m_1, \ldots, m_N; w_1, \ldots, w_r]\!]$ be admissible with respect to the variables $q_1, \ldots, q_d$. Suppose F is symmetric in $w_1, \ldots, w_r$, in $m_1, \ldots, m_N$, and symmetric in $q_1, \ldots, q_d$, then there exist power series $G_{\mu,\nu,\xi}(Q)$ labeled by partitions μ, ν, ξ, such that*

$$F(Q; \vec{m}; \vec{w}; e^{\lambda_1}, \ldots, e^{\lambda_n}) = \prod_{\mu,\nu} G_{\mu,\nu,\xi}(Q)^{\int_Y c_\nu(Y) c_\mu(V) c_\xi(E)}.$$

3. Segre and Verlinde invariants on $\mathbb{C}^2$

3.1. **An auxiliary invariant for compact surfaces.** A general tactic for studying the Segre and Verlinde series is using a more general genus. In the non-virtual surface case this could be [GM22, Equation (1.1)] defined below. In the virtual case we will use the invariant (3.11). For Calabi-Yau 4-folds, we consider the Nekrasov genus (5.2), introduced by [NP19].

For a vector bundle V over Y define

$$\Lambda_z(V) = \sum_{i \geq 0} [\Lambda^i V] z^i \in K^0(Y)[z], \quad \Lambda_z(-V) = \sum_{i \geq 0} [\operatorname{Sym}^i V](-z)^i \in K^0(Y)[\![z]\!]$$

which extends to a homomorphism $\Lambda_z : (K^0(Y), +) \to (K^0(Y)[\![z]\!], \cdot)$. For $\alpha \in K^0(S)$, set

$$I(\alpha; q, z) := \sum_{n=0}^{\infty} (-q)^n \chi\left(\operatorname{Hilb}^n(S), (\Lambda_{-z} \alpha^{[n]}) \otimes \det(\mathcal{O}_S^{[n]})^{-1}\right). \tag{3.1}$$

This invariant is chosen so that the Chern series and Verlinde series can be recovered from it by taking limits. L. Göttsche and A. Mellit computed some of its universal series and used them to find universal series for the Segre and Verlinde invariants. We state their theorem for the case of rank $r = 2$ for later use.

Theorem 3.1 (Göttsche-Mellit [GM22])**.** *For any $r \in \mathbb{Z}$, there exist power series $G_0, G_1, G_2, G_3, G_4 \in \mathbb{Z}[\![q, z]\!]$ such that for all smooth projective surfaces S and $\alpha \in K^0(S)$ of rank r, we have*

$$I(\alpha; q, z) = G_0(q, z)^{c_2(\alpha)} G_1(q, z)^{\chi(\det \alpha)} G_2(q, z)^{\frac{1}{2}\chi(\mathcal{O}_S)} G_3(q, z)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} G_4(q, z)^{K_S^2}.$$

When $r = 2$, we have

$$\begin{aligned} &G_0(q, z) = 1 - qz, \quad G_1(q, z) = \frac{1 - qz^2}{1 - qz}, \quad G_2(q, z) = \frac{(1-q)^2}{(1-qz)^2} \\ &G_3(q, z) = G_4(q, z) = 1. \end{aligned}$$

3.2. **Equivariant invariants on Hilbert schemes.** As mentioned in the introduction, the equivariant invariants on $S = \mathbb{C}^2$ are defined by equivariant localization. The following definition gives a more precise description. Let $\mathsf{T} = (\mathbb{C}^2)^*$ be the 2-dimensional torus acting on $S = \mathbb{C}^2$ by scaling:

$$(t_1, t_2) \cdot (x_1, x_2) = (t_1 x_1, t_2 x_2) \in \mathbb{C}^2.$$

This lifts to an action on $\mathrm{Hilb}^n(S)$. For a T-equivariant bundle V, $V^{[n]}$ is also T-equivariant. The T-fixed locus on $\mathrm{Hilb}^n(S)$ is a finite collection of reduced points corresponding to monomial ideals of $\mathbb{C}[x_1, x_2]$ by Lemma 3.3. Therefore we can define the equivariant Segre and Chern invariants by replacing the usual integration with equivariant integration (2.1). Similarly, the Verlinde invariants on S are defined using K-theoretic equivariant localization [Tho92, Theorem 3.5]. Since we are interested in comparing the Segre and Verlinde series, we convert the Verlinde invariants into cohomological invariants using the equivariant Hirzebruch-Riemann-Roch formula [EG99, Corollary 3.1].

Definition 3.2. When $S = \mathbb{C}^2$, the *equivariant Chern series* of $\alpha \in K_{\mathsf{T}}(S)$ is

$$\begin{aligned} I^{\mathcal{C}}(\alpha; q) &:= \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} c(\alpha) \\ &:= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Hilb}^n(S)^{\mathsf{T}}} \frac{c(\alpha^{[n]}|_Z)}{e(T_Z)} \\ &\in H^*_{\mathsf{T}}(\mathrm{pt})_{\mathrm{loc}}[\![q]\!] = \mathbb{C}(\lambda_1, \lambda_2)[\![q]\!] \end{aligned}$$

where T_Z is the Zariski tangent space of $\mathrm{Hilb}^n(S)$ at $[Z]$. The *equivariant Verlinde number* is

$$\begin{aligned} I^{\mathcal{V}}(\alpha; q) &:= \sum_{n=0}^{\infty} q^n \int_{\mathrm{Hilb}^n(S)} \mathrm{ch}(\det(\alpha^{[n]}))\, \mathrm{td}(T) \\ &:= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Hilb}^n(S)^{\mathsf{T}}} \frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))\, \mathrm{td}(T_Z)}{e(T_Z)} \\ &= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Hilb}^n(S)^{\mathsf{T}}} \frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))}{\mathrm{ch}(\Lambda_{-1} T_Z^\vee)} \in \left(\prod_{i=0}^{\infty} H^i_{\mathsf{T}}(\mathrm{pt}) \right)_{\mathrm{loc}} [\![q]\!]. \end{aligned}$$

Lemma 3.3. *The T-fixed locus $\mathrm{Hilb}^n(S)^{\mathsf{T}}$ consists of finitely many reduced points.*

Proof. Since T acts on $\mathbb{C}^2$ by scaling coordinates, it acts on the coordinate ring $\mathbb{C}[x_1, x_2]$ by

$$(t_1, t_2) \cdot x_i = t_i^{-1} x_i.$$

Observe the T-fixed ideals of $\mathbb{C}[x_1, x_2]$ are exactly the monomial ideals, of which there are only finitely many that have length n. For these points to be reduced, it suffices to show the Zariski tangent space in $\mathrm{Hilb}^n(S)$ has no T-fixed parts, which follows from the characterization of the tangent space (3.3). □

We shall explain how to calculate the above invariants. First consider the correspondence between monomial ideals of $\mathbb{C}[x_1, x_2]$ and partitions. For a partition μ, the corresponding point $[Z_\mu] \in \mathrm{Hilb}^n(S)$ is given by the monomial ideal I_{Z_μ} such that

$$\mathcal{O}_{Z_\mu} = \mathbb{C}[x_1, x_2]/I_{Z_\mu} = \mathrm{span}\{x_1^{c(\square)} x_2^{r(\square)} : \square \in \mu\}.$$

It follows that the character of $\mathcal{O}_{Z_\mu}$ is

$$\sum_{\square \in \mu} t_1^{-c(\square)} t_2^{-r(\square)} \in K_{\mathsf{T}}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm}, t_2^{\pm}].$$

Let $V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$ be a equivariant bundle on S of rank r, twisted with weights v_i for $i = 1, \ldots, r$. For any point $[Z_\mu] \in \mathrm{Hilb}^n(X)^{\mathsf{T}}$, the fiber of $V^{[n]}$ is

$$V^{[n]}|_{Z_\mu} = p_*(\mathcal{O}_{\mathcal{Z}} \otimes q^* V)|_{Z_\mu} = H^0(X, \mathcal{O}_{Z_\mu} \otimes V) = H^0(V|_{Z_\mu}).$$

The right hand side is the following rn-dimensional representation in $K_{\mathsf{T}}(\mathrm{pt})$:

$$\bigoplus_{i=1}^r \mathcal{O}_{Z_\mu}\langle v_i\rangle = \sum_{i=1}^r \sum_{\square \in \mu} v_i t_1^{-c(\square)} t_2^{-r(\square)}. \tag{3.2}$$

It was shown in [ES87, Lemma 3.2] that the Zariski tangent bundle at $[Z_\mu]$ is

$$T_{Z_\mu} = \sum_{\square \in \mu} t_1^{a(\square)+1} t_2^{-l(\square)} + t_1^{-a(\square)} t_2^{l(\square)+1}. \tag{3.3}$$

Denote $w_i := c_1^{\mathsf{T}}(v_i)$ the equivariant Chern roots of V, we may now expand Definition 3.2 and get

$$I^{\mathcal{C}}(V; q) = \sum_{\mu} q^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{i=1}^r (1 + w_i - c(\square)\lambda_1 - r(\square)\lambda_2)}{((a(\square)+1)\lambda_1 - l(\square)\lambda_2)((l(\square)+1)\lambda_2 - a(\square)\lambda_1)},$$

$$I^{\mathcal{V}}(V; q) = \sum_{\mu} q^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{i=1}^r v_i t_1^{-c(\square)} t_2^{-r(\square)}}{\left(1 - t_1^{-(a(\square)+1)} t_2^{l(\square)}\right)\left(1 - t_1^{a(\square)} t_2^{-(l(\square)+1)}\right)}.$$

Note the expression for the Verlinde series uses the identification in Remark 2.4.

An important tool for studying the Segre and Verlinde series on $S = \mathbb{C}^2$ used by [GM22] is the *master partition function.* For each $r \in \mathbb{Z}$, it is defined by

$$\Omega(Q; z_1, \ldots, z_r; q_1, q_2) := \sum_{\mu} Q^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{i=1}^r (1 - q_1^{c(\square)} q_2^{r(\square)} z_i)}{(q_1^{a(\square)+1} - q_2^{l(\square)})(q_1^{a(\square)} - q_2^{l(\square)+1})}.$$

On $S = \mathbb{C}^2$ with the bundle $V = \oplus_{i=1}^r \mathcal{O}_S\langle w_i\rangle$, the invariant (3.1) can be defined equivariantly by

$$I(V; q, z) = \Omega(q; z e^{w_1}, \ldots, z e^{w_r}; e^{-\lambda_1}, e^{-\lambda_2}).$$

Furthermore, using the explicit expressions of $I^{\mathcal{C}}$ and $I^{\mathcal{V}}$ above, one can show that they are specializations of Ω as follows.

Proposition 3.4. ([GM22, Proposition 3.5]) *The Chern and Verlinde series satisfy the following limits:*

$$I^{\mathcal{C}}(V;q) = \lim_{\varepsilon\to 0}\Omega\left(-q\varepsilon^{2-r}(1+\varepsilon)^r;\frac{e^{-\varepsilon w_1}}{1+\varepsilon},\dots,\frac{e^{-\varepsilon w_r}}{1+\varepsilon};e^{\varepsilon\lambda_1},e^{\varepsilon\lambda_2}\right),$$
$$I^{\mathcal{V}}(V;q) = \lim_{\varepsilon\to 0}\Omega\left((-1)^r q\varepsilon^{r+1};\varepsilon^{-1}e^{w_1},\dots,\varepsilon^{-1}e^{w_r},\varepsilon^{-1};e^{-\lambda_1},e^{-\lambda_2}\right).$$

3.3. **Relation to projective toric surfaces.** We consider what the equivariant invariants will be for a projective toric surface S' with a natural action by the torus $\mathsf{T}=(\mathbb{C}^*)^2$, and compare them with the case $S=\mathbb{C}^2$. More details on this reduction can be found in [GM22, Section 3.2]; see also [Arb21, Section 6.2] and [LYZ02, Section 3.2].

Let the fixed points on S' be $p_1,\dots,p_M$. Denote the Chern roots of the tangent space at p_i by $a_1^{(i)},a_2^{(i)}$. Let V' be a T-equivariant bundle of rank r on S' with Chern roots $w_1^{(i)},\dots,w_r^{(i)}$ at p_i. By equivariant localization, (3.1) for S' and V' can be expressed as

$$I(V';q,z) = \left(\prod_{i=1}^{M}\Omega(q;ze^{w_1^{(i)}},\dots,ze^{w_r^{(i)}};e^{-a_1^{(i)}},e^{-a_2^{(i)}})\right)\Bigg|_{\lambda_1=\lambda_2=0}. \tag{3.4}$$

As remarked in [EG95b], since S' is compact, the product on the right hand side lives in $H^*_{\mathsf{T}}(\mathrm{pt})=\mathbb{C}[\lambda_1,\lambda_2]$. Therefore it is indeed valid to set $\lambda_1=\lambda_2=0$, and the equality follows from Bott residue formula. This helps us in finding universal series for the $S=\mathbb{C}^2$ case because the universal series on the left hand side is already known by Theorem 3.1.

Using Macdonald polynomials and results from [Mel18], Göttsche and Mellit [GM22, Proposition 2.5] showed that Ω is admissible with respect to q_1,q_2 in the sense of Definition 2.6. Applying expansion (2.3), we have

$$\begin{aligned}&\log\Omega(q;ze^{w_1},\dots,ze^{w_r};e^{-\lambda_1},e^{-\lambda_2})\\ &=\sum_{\substack{\mu\text{ partition}\\ k_1,k_2\geq -1}} H_{\mu,k_1,k_2}(q,z)\cdot\int_S E_{k_1,k_2}(c_1^{\mathsf{T}}(S),c_2^{\mathsf{T}}(S))c_\mu^{\mathsf{T}}(V)\end{aligned} \tag{3.5}$$

for some series H_{μ,k_1,k_2}. Note that the integrand on the right is a homogeneous rational function in the variables λ_1,λ_2 of total degree $|\mu|+k_1+k_2$. Exponentiating both sides, we get

$$\Omega(q;ze^{w_1},\dots,ze^{w_r};e^{-\lambda_1},e^{-\lambda_2}) = \prod_{\substack{\mu\text{ partition}\\ k_1,k_2\geq -1}} H_{\mu,k_1,k_2}(q,z)^{\int_S E_{k_1,k_2}(c_1^{\mathsf{T}}(S),c_2^{\mathsf{T}}(S))c_\mu^{\mathsf{T}}(V)}$$

Substituting this into (3.4) yields

$$\begin{aligned}I(V';q,z) &= \left(\prod_{\mu,k_1,k_2\geq -1} H_{\mu,k_1,k_2}(q,z)^{\int_{S'} E_{k_1,k_2}(c_1^{\mathsf{T}}(S'),c_2^{\mathsf{T}}(S'))c_\mu^{\mathsf{T}}(V')}\right)\Bigg|_{\lambda_1=\lambda_2=0}\\ &= \prod_{|\mu|+k_1+k_2=0} H_{\mu,k_1,k_2}(q,z)^{\int_{S'} E_{k_1,k_2}(c_1(S'),c_2(S'))c_\mu(V')}.\end{aligned}$$

where the integral in the first line is equivariant integration, while the integral in the second line is the usual non-equivariant integration. Comparing this expansion with the one in Theorem 3.1, together with a quick computation that $E_{-1,-1}(x_1,x_2)=1, E_{-1,0}(x_1,x_2)=\frac{1}{2}x_1, E_{0,0}(x_1,x_2)=$

$x_2, E_{-1,1}(x_1,x_2) = \frac{1}{2}(x_1^2 - 2x_2)$, we obtain [GM22, Equation (3.16)]

$$\begin{aligned}
&\log G_0(q,z) = H_{(2),-1,-1}(q,z), \quad \log G_1(q,z) = 2H_{(1,1),-1,-1}(q,z),\\
&\log G_2(q,z) = 24(H_{(0),0,0}(q,z) - 2H_{(0),-1,1}(q,z)) - 4H_{(1,1),-1,-1}(q,z),\\
&\log G_3(q,z) = H_{(1),-1,0}(q,z) + H_{(1,1),-1,-1}(q,z),\\
&\log G_4(q,z) = -H_{(0),0,0}(q,z) + 3H_{(0),-1,1}(q,z) - \frac{1}{2}(H_{(1),-1,0}(q,z) + H_{(1,1),-1,-1}(q,z))
\end{aligned}$$

The series $G_i(q,z)$ are as in Theorem 3.1, which give, in rank $r=2$,

$$\begin{aligned}
&H_{(2),-1,-1}(q,z) = \log(1-qz),\\
&H_{(1,1),-1,-1}(q,z) = -H_{(1),-1,0}(q,z) = \frac{1}{2}\log\frac{1-qz^2}{1-qz},\\
&H_{(0),0,0}(q,z) = 3H_{(0),-1,1}(q,z) = \frac{1}{4}\log\frac{(1-q)(1-qz^2)}{(1-qz)^2}.
\end{aligned} \tag{3.6}$$

To summarize, we have observed that the universal series for the projective case are exactly the ones for the $S=\mathbb{C}^2$ case whose powers have degree 0 in $H^*_{\mathsf{T}}(\mathrm{pt})$.

3.4. **Virtual invariants.** Before defining virtual invariants, we recall the notion of a perfect obstruction theory in the sense of [BF98, Definition 4.4]. For our purposes, we use the following simplified version.

Definition 3.5. Let X be a scheme over $\mathbb{C}$. An *obstruction theory* is a complex of vector bundles

$$E^\bullet = [\ldots \to E^{-2} \to E^{-1} \to E^0]$$

for some $a \in \mathbb{Z}$, together with a morphism in the derived category $D(\mathrm{QCoh}(X))$ to the cotangent complex

$$\varphi : E^\bullet \to L_X^\bullet$$

such that $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective. It is a *(2-term) perfect obstruction theory* if $E^i = 0$ for $i \neq 0, -1$. The *virtual tangent space* $T^{\mathrm{vir}} = E_\bullet = (E^\bullet)^*$ is the class of the dual complex of a given obstruction theory.

Let S be a surface and E a torsion free sheaf. It is well known that $\mathrm{Quot}_S(E,n)$ admits an obstruction theory given by the dual complex of $\mathbf{R}\mathscr{H}om_p(\mathcal{I},\mathcal{F})$, where $\mathcal{I},\mathcal{F}$ are respectively the universal subsheaf and quotient sheaf. When S is a projective surface, [MOP15, Lemma 1] shows that this obstruction theory is perfect of virtual dimension nN. Using this, we can define a virtual fundamental class $[\mathrm{Quot}_S(E,n)]^{\mathrm{vir}}$ via the methods from [BF98, LT96] as well as a virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$ using [CFK09]. Applying the same argument for $S=\mathbb{C}^2$ gives us a T-equivariant perfect obstruction theory. We note that since $\mathcal{F}$ is compactly supported, the Ext-groups are finite dimensional vector spaces, so the steps involving Serre duality still work.

Let $S=\mathbb{C}^2$ and $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$. Recall from the introduction the following tori:

$$\mathsf{T}_0 = (\mathbb{C}^*)^2, \quad \mathsf{T}_1 = (\mathbb{C}^*)^N, \quad \mathsf{T}_2 = (\mathbb{C}^*)^{r+s}.$$

Set $\mathsf{T} = \mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2$, with

$$\begin{aligned}
K_{\mathsf{T}}(\mathrm{pt}) &= \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}; y_1^{\pm 1}, \ldots, y_N^{\pm 1}; v_1^{\pm 1}, \ldots, v_{r+s}^{\pm 1}],\\
H^*_{\mathsf{T}}(\mathrm{pt}) &= \mathbb{C}[\lambda_1,\lambda_2; m_1,\ldots,m_N; w_1,\ldots,w_{r+s}].
\end{aligned}$$

Under these actions, the T_1-fixed locus of $\mathrm{Quot}_S(E,n)$ decomposes into the form

$$0 \to \oplus_{i=1}^N I_i\langle y_i\rangle \to \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle \to \oplus_{i=1}^N F_i\langle y_i\rangle \to 0.$$

Thus the T_1-fixed locus can be identified as

$$\bigsqcup_{n_1+\cdots+n_N=n} \mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_N}(S).$$

Consequently, the T-fixed locus $\mathrm{Quot}_S(E,n)^{\mathsf{T}}$ consists of finitely many reduced points of form

$$Z_\mu=([Z_1],[Z_2],\dots,[Z_N])\in\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_N}(S),$$

labeled by N-colored partitions $\mu=\left(\mu^{(1)},\dots,\mu^{(N)}\right)$.

The following equivariant invariants are defined similarly to Definition 3.2, now motivated by *virtual* equivariant localization [GP97], *virtual* K-theoretic equivariant localization [CFK09, Theorem 5.3.1], and the *virtual* Hirzebruch-Riemann-Roch formula [RS21, Corollary 1.2].

Definition 3.6. Let $S=\mathbb{C}^2$ and

$$\alpha=[\oplus_{i=1}^r\mathcal{O}_Y\langle v_i\rangle]-[\oplus_{i=r+1}^{r+s}\mathcal{O}_Y\langle v_i\rangle]\in K_{\mathsf{T}}(S)$$

the *equivariant virtual Segre, Chern, Verlinde series* on Quot schemes are respectively

$$\mathcal{S}_S(E,\alpha;q):=\sum_{n=0}^\infty q^n\sum_{Z\in\mathrm{Quot}_S(E,n)^{\mathsf{T}}}\frac{s(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{vir}})},$$

$$\mathcal{C}_S(E,\alpha;q):=\sum_{n=0}^\infty q^n\sum_{Z\in\mathrm{Quot}_S(E,n)^{\mathsf{T}}}\frac{c(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{vir}})},$$

$$\mathcal{V}_S(E,\alpha;q):=\sum_{n=0}^\infty q^n\sum_{Z\in\mathrm{Quot}_S(E,n)^{\mathsf{T}}}\frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))}{\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{vir}})^\vee)}.$$

We shall describe how to calculate these invariants, and refer to [FMR21, Section 5.1] and [Lim21, Section 3.3] for the following argument. On each T_1-fixed locus

$$D=\mathrm{Hilb}^{n_1}(S)\times\cdots\times\mathrm{Hilb}^{n_N}(S),$$

the universal subsheaf and universal quotient sheaf of D are $\bigoplus_{i=1}^N I_{\mathcal{Z}_i}\langle y_i\rangle$ and $\bigoplus_{j=1}^N\mathcal{O}_{\mathcal{Z}_j}\langle y_i\rangle$ where $\mathcal{Z}_i$ is the universal subscheme of $\mathrm{Hilb}^{n_i}(S)$. The virtual tangent bundle over D is then

$$T_D^{\mathrm{vir}}=\bigoplus_{i,j=1}^N\mathbf{R}\mathscr{H}om_p(I_{\mathcal{Z}_i},\mathcal{O}_{\mathcal{Z}_j})\langle y_i^{-1}y_j\rangle$$

where $p:D\times X\to D$ is the projection. Further restricting to each $Z_\mu=([Z_1],[Z_2],\dots,[Z_N])\in\mathrm{Quot}_S(E,n)^{\mathsf{T}}$ gives the virtual tangent bundle at Z_μ as follows

$$T_{Z_\mu}^{\mathrm{vir}}=\bigoplus_{i,j=1}^N\mathrm{Ext}(I_{Z_i},\mathcal{O}_{Z_j})\langle y_i^{-1}y_j\rangle\in K_{\mathsf{T}}(S).\tag{3.7}$$

To give an explicit formula for T^{vir}, we consider a T_0-equivariant free resolution of I_{Z_i}. We refer to [Eis95, Page 439] for the following *Taylor resolution.* Say I_{Z_i} is generated by monomials $m_1,\dots,m_s$. For each $k=0,\dots,s$, let F_k be the free $\mathbb{C}[x_1,\dots,x_n]$-module module with basis $\{e_I\}$, indexed by subsets $I\subseteq\{1,\dots,s\}$ of size k. Set

$$m_I=\text{least common multiple of }\{m_i:i\in I\}.$$

For $k=1,\dots,s$, define differential $d_k:F_k\to F_{k-1}$ by

$$d_k(e_I)=\sum_{j=1}^k(-1)^j\frac{m_I}{m_{I-\{i_j\}}}e_{I-\{i_j\}}$$

for each subset $I = \{i_1, \dots, i_k\}$ such that $i_1 < \cdots < i_k$. Giving each e_I the weight of m_I, we obtain the T_0-equivariant free resolution

$$0 \to F_s \to \dots \to F_0 \to I_{Z_i} \to 0$$

where

$$F_k = \bigoplus_{I \subseteq \{1,\dots,s\}, |I|=k} \mathcal{O}_S \langle m_I(t) \rangle$$

for some $d_{kI} \in \mathbb{Z}^2$. Define

$$P(I_{Z_i}) = \sum_{k,|I|=k} (-1)^k m_I(t). \tag{3.8}$$

Note that the character of $\mathcal{O}_S = \mathbb{C}[x_1, x_2]$ is $\sum_{i,j \geq 0} t_1^{-i} t_2^{-j} = 1/(1 - t_1^{-1})(1 - t_2^{-1})$, so the character of $\mathcal{O}_{Z_i} = \mathcal{O}_S / I_{Z_i}$ is

$$Q_i := \frac{1 - P(I_{Z_i})}{(1 - t_1^{-1})(1 - t_2^{-1})}.$$

Therefore the character of $T^{\mathrm{vir}}_{Z_\mu}$ in $K_{\mathsf{T}}(\mathrm{pt})$ can be expressed as

$$\begin{aligned}
\bigoplus_{i,j=1}^{N} \operatorname{Ext}(I_{Z_i}, \mathcal{O}_{Z_j}) \langle y_i^{-1} y_j \rangle &= \sum_{i,j=1}^{N} \sum_{k,|I|=k} (-1)^k \operatorname{Hom}(\mathcal{O}_S \langle m_I(t) \rangle, \mathcal{O}_{Z_j}) y_i^{-1} y_j \\
&= \sum_{i,j=1}^{N} \sum_{k,I} (-1)^k \mathcal{O}_{Z_j} \langle m_I(t)^{-1} \rangle y_i^{-1} y_j \\
&= \sum_{i,j=1}^{N} \overline{P(I_{Z_i})} Q_j y_i^{-1} y_j \\
&= \sum_{i,j=1}^{N} (Q_j - (1 - t_1)(1 - t_2) \overline{Q_i} Q_j) \cdot y_i^{-1} y_j
\end{aligned} \tag{3.9}$$

where $\overline{(\cdot)}$ denotes the involution $t_i \mapsto t_i^{-1}$. For the T-equivariant bundle $V = \oplus_{i=1}^r \mathcal{O}_S \langle v_i \rangle$, the fiber of $V^{[n]}$ over $Z_\mu = (Z_1, \dots Z_N)$ is the rn-dimensional representation

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{N} \mathcal{O}_{Z_j} \langle v_i y_j \rangle = \sum_{i=1}^{r} \sum_{j=1}^{N} \sum_{\square \in \mu^{(j)}} v_i y_j t_1^{-c(\square)} t_2^{-r(\square)}.$$

Substituting the above calculations into the definition, we obtain the following expressions for the Chern and Verlinde series of vector bundles

$$\begin{aligned}
\mathcal{C}_S(E, V; q) &:= \sum_{\mu} q^{|\mu|} \frac{\prod_{j=1}^{N} \prod_{\square \in \mu^{(j)}} \prod_{i=1}^{r} (1 + w_i + m_j - c(\square)\lambda_1 - r(\square)\lambda_2)}{e(T^{\mathrm{vir}}|_{Z_\mu})}, \\
\mathcal{V}_S(E, V; q) &:= \sum_{\mu} q^{|\mu|} \frac{\prod_{j=1}^{N} \prod_{\square \in \mu^{(j)}} \prod_{i=1}^{r} v_i y_j t_1^{-c(\square)} t_2^{-r(\square)}}{\operatorname{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^{\vee})}.
\end{aligned} \tag{3.10}$$

3.5. **Universal series expansion.** For projective surfaces, define an auxiliary virtual invariant (c.f. (3.1)) as follows

$$\mathcal{N}_S(E,\alpha;q,z) := \sum_{n=0}^{\infty} q^n \chi^{\mathrm{vir}} \left(\mathrm{Quot}_S(E,n), \Lambda_{-z}\alpha^{[n]} \right). \tag{3.11}$$

where z is considered as the weight of an extra $\mathbb{C}^*$-action that is trivial on S and $\mathrm{Quot}_S(E,n)$. We shall refer to this as the *Nekrasov genus* for Quot schemes of surfaces (c.f. (5.2)).

Similar to before, we generalize this to the equivariant setting using virtual equivariant localization. On $S = \mathbb{C}^2$ for vector bundles, this is given by:

$$\begin{aligned}\mathcal{N}_S(E,V;q,z) &:= \sum_{\mu} q^{|\mu|} \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j z)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)} \\ &\in \mathbb{Q}(t_1,t_2;y_1,\dots,y_N)[\![q,z]\!].\end{aligned} \tag{3.12}$$

The choice of this invariant is based on Göttsche-Mellit's invariant (3.1). The following Chern and Verlinde limits are satisfied, analogous to [GM22, Proposition 3.5].

Lemma 3.7. *For $S = \mathbb{C}^2$, the Chern series and the Verlinde series can be retrieved from $\mathcal{N}_S$ by taking limits. We have*

$$\mathcal{C}_S(E,V;q) = \lim_{\varepsilon\to 0} \mathcal{N}_S \left(E,V;(-1)^N q\varepsilon^{(N-r)}(1+\varepsilon)^r, (1+\varepsilon)^{-1} \right) |_{\vec{\lambda}\rightsquigarrow -\varepsilon\vec{\lambda}, \vec{w}\rightsquigarrow -\varepsilon\vec{w}, \vec{m}\rightsquigarrow -\varepsilon\vec{m}},$$
$$\mathcal{V}_S(E,V;q) = \lim_{\varepsilon\to 0} \mathcal{N}_S \left(E,V;(-1)^r q\varepsilon^r, \varepsilon^{-1} \right).$$

Proof. For the Chern limit, first consider the substitutions $\lambda_i \rightsquigarrow -\varepsilon\lambda_i, w_i \rightsquigarrow -\varepsilon w_i, m_i \rightsquigarrow -\varepsilon m_i$. This turns the term $\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - t_1^{-c(\Box)} t_2^{-r(\Box)} v_i y_j (1+\varepsilon)^{-1})$ into

$$\begin{aligned}&\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r 1 - \frac{e^{-\varepsilon(w_i+m_j-c(\Box)\lambda_1-r(\Box)\lambda_2)}}{1+\varepsilon} \\ =&\frac{1}{(1+\varepsilon)^{r|\mu|}} \prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1+\varepsilon - e^{-\varepsilon(w_i+m_j-c(\Box)\lambda_1-r(\Box)\lambda_2)}) \\ =&\left(\frac{\varepsilon}{1+\varepsilon}\right)^{r|\mu|} \prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon))\end{aligned}$$

For the denominator in the sum (3.12), we note that for a Chern root x, substituting it by $-\varepsilon x$ turns $1 - e^{-x} = x - \frac{x^2}{2} + ...$ into $1 - e^{\varepsilon x} = -\varepsilon(x + O(\varepsilon))$. Therefore after the substitution, the denominator $\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)$ becomes

$$(-1)^{N|\mu|}\varepsilon^{N|\mu|}(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon))$$

Substituting back into (3.12), the Chern limit becomes the limit of

$$\begin{aligned}&\sum_\mu (-1)^{N|\mu|} q^{|\mu|} \varepsilon^{(N-r)|\mu|} (1+\varepsilon)^{r|\mu|} \cdot \frac{\varepsilon^{r|\mu|}}{(-1)^{N|\mu|}\varepsilon^{N|\mu|}(1+\varepsilon)^{r|\mu|}} \cdot \\ &\frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon))}{(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon))} \\ =&\sum_\mu q^{|\mu|} \frac{\prod_{j=1}^N \prod_{\Box\in\mu^{(j)}} \prod_{i=1}^r (1 - c(\Box)\lambda_1 - r(\Box)\lambda_2 + w_i + m_j + O(\varepsilon))}{(e(T^{\mathrm{vir}}_{Z_\mu}) + O(\varepsilon))}\end{aligned}$$

which converges to $\mathcal{C}_S(E,V;q)$ by (3.10). For the Verlinde series, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathcal{N}_S(E,V;(-1)^r q\varepsilon^r,\varepsilon^{-1})\\
=&\lim_{\varepsilon\to 0}\sum_{\mu}(-1)^{r|\mu|}q^{|\mu|}\varepsilon^{r|\mu|}\cdot\frac{\prod_{j=1}^N\prod_{\Box\in\mu^{(j)}}t_1^{c(\Box)}t_2^{r(\Box)}\prod_{i=1}^r(1-t_1^{-c(\Box)}t_2^{-r(\Box)}v_iy_j\varepsilon^{-1})}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}\\
=&\lim_{\varepsilon\to 0}\sum_{\mu}q^{|\mu|}\frac{\prod_{j=1}^N\prod_{\Box\in\mu^{(j)}}\prod_{i=1}^r(t_1^{-c(\Box)}t_2^{-r(\Box)}v_iy_j-\varepsilon)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}\\
=&\mathcal{V}_S(E,V;q).
\end{aligned}$$

□

Before starting the proof for universal series expressions, let us discuss how the expansion of $\mathcal{N}_S(E,V;q,z)$ as a formal Laurent series in the variables $\vec{\lambda},\vec{m},\vec{w},q,z$ would look like. In [Arb21, Proposition 3.2], N. Arbesfeld shows that invariants such as $[q^n]\mathcal{N}_S(E,V;q,z)$ can be written as a quotient whose numerator is a Laurent polynomial in $\vec{t},\vec{y},\vec{v},z$, and whose denominator is of the form $\prod_{\mathsf{w}}(1-\mathsf{w})$ for some *non-compact weights* w in the sense of the following definition.

Definition 3.8. [Arb21, Definition 3.1] Let M be a quasi-projective scheme with an action by some torus T. For a weight $\mathsf{w}\in\mathsf{T}^\vee$, denote T_{w} the maximal subtorus of T containing $\ker\mathsf{w}$. If the fixed locus $M^{\mathsf{T}_{\mathsf{w}}}$ is proper, then w is a *compact weight*, otherwise, it is a *non-compact weight*.

Fasola-Monavari-Ricolfi used this to prove that the K-theoretic Donaldson-Thomas partition functions on $\mathbb{C}^3$ are Laurent polynomials with respect to the variables $y_1,\dots,y_N$ [FMR21, Theorem 6.5]. We give an outline of their argument, applied to the invariant $\mathcal{N}_S$ for $S=\mathbb{C}^2$. First note that by (3.9), for any N-colored partition μ, we have

$$\frac{1}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{vir}}|_{Z_\mu})^\vee)}=A(\vec{t})\prod_{1\le i,j\le N,i\ne j}\frac{\prod_{a\in A_{ij}}(1-y_i^{-1}y_jt^a)}{\prod_{b\in B_{ij}}(1-y_i^{-1}y_jt^b)}$$

for some series $A(\vec{t})\in\mathbb{Q}[\![t_1,t_2]\!]_{\mathrm{loc}}$ and some sets of weights A_{ij},B_{ij}. We shall show that the denominator of $\mathcal{N}_S$ does not have factors of the form $(1-y_i^{-1}y_jt^b)$ for any $i\ne j$ and $b\in\mathbb{Z}^2$. By [Arb21, Proposition 3.2], we need to prove $\mathsf{w}=y_i^{-1}y_jt^b$ is a compact weight. Since

$$\ker\mathsf{w}=\{(\vec{t},\vec{y},\vec{v}):y_i=y_jt^b\}$$

is itself a torus, we have $\mathsf{T}_{\mathsf{w}}=\ker\mathsf{w}$. By definition, it suffices to show $\mathrm{Quot}_S(E,n)^{\mathsf{T}_{\mathsf{w}}}$ is proper. With the automorphism $\mathsf{T}\to\mathsf{T}$ defined by

$$(\vec{t},y_1,\dots,y_j,\dots,y_N,\vec{v})\mapsto(\vec{t},y_1,\dots,y_jt^b,\dots,y_N,\vec{v}),$$

we identify the subgroup T_{w} to $\mathsf{T}_0\times\{(w_1,\dots,w_N):w_i=w_j\}\times\mathsf{T}_2$, which contains the subgroup $\mathsf{T}_0=\mathsf{T}_0\times\{(1,\dots,1)\}$. This gives us an inclusion

$$\mathrm{Quot}_S(E,n)^{\mathsf{T}_{\mathsf{w}}}\hookrightarrow\mathrm{Quot}_S(E,n)^{\mathsf{T}_0}.$$

The quotients in the fixed locus $\mathrm{Quot}_S(E,n)^{\mathsf{T}_0}$ are all supported at the origin $0\in\mathbb{C}^2$, so the fixed locus lies inside the punctual Quot scheme $\mathrm{Quot}_S(E,n)_0$. The punctual Quot scheme is proper since it is a fiber of the Quot-to-Chow map $\mathrm{Quot}_S(E,n)\to\mathrm{Sym}^n S$, which is a proper morphism [FMR21, Remark 3.4]. In conclusion, $[q^n]\mathcal{N}_S(E,V;q,z)$ is a Laurent polynomial with respect to the variables $y_1,\dots,y_N$, so it can be expanded into a power series with respect to the cohomological parameters $m_1,\dots,m_N$.

Furthermore, if w is a weight that contains both t_1 and t_2, then we have $\mathsf{T}_{\mathsf{w}}\cong\{(t_1,t_2):t_1t_2=1\}\times\mathsf{T}_1\times\mathsf{T}_2$. The fixed locus of this subgroup remains the same as that of T, as explained in the next section for reduced invariants. Therefore w is a compact weight, and the denominator of $\mathcal{N}_S$ will not contain factors of the form $(1 - t_1^a t_2^b)$ for any $a \neq 0$, $b \neq 0$. This means in cohomology, $[q^n]\mathcal{N}_S(E, V; q, z)$ can be expanded into a Laurent series in λ_1, λ_2 whose coefficients are power series in $\vec{m}, \vec{w}, z$, where the degrees on λ_1, λ_2 are bounded below individually. We shall see the importance of this lower bound in the proof of the following theorem.

Theorem 3.9. *Let $S = \mathbb{C}^2$. For any $r \in \mathbb{Z}$, $N > 0$, there exist universal power series $A_{\mu,\nu,\xi}(q), B_{\mu,\nu,\xi}(q)$, dependent on N and r, such that for $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha \in K_{\mathsf{T}}(S)$ of rank r, the equivariant virtual Segre and Verlinde series on $\mathrm{Quot}_S(E, n)$ can be written as the following infinite products*

$$\mathcal{S}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{V}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)},$$

$$\mathcal{C}_S(E, \alpha; q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)c_1(S)}.$$

Proof. We begin with the case where α is a vector bundle V. Assume $V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$, and at end of the proof, we can generalize this to arbitrary T-equivariant bundles by substituting T-weights into the variables $v_1, \dots, v_r$.

Begin by expanding $\log \mathcal{N}_S(E, V; q, z)$ as a Laurent series in λ_1, λ_2 as follows:

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{(j,k)\in\mathbb{Z}^2} H_{j,k}(q, z; \vec{m}; \vec{w})\lambda_1^j \lambda_2^k$$

for some series $H_{j,k} \in \mathbb{Q}[\![q, z; m_1, \dots, m_N; w_1, \dots, w_r]\!]$. By the symmetry in $w_1, \dots, w_r$ and the symmetry in $m_1, \dots, m_N$, this expands to

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{\substack{\mu,\xi \text{ partitions}\\ j,k\geq -1}} G_{\mu,\xi,j,k}(q, z) \cdot \lambda_1^j \lambda_2^k c_\mu(V)c_\xi(E) + \sum_{\substack{\mu,\xi \text{ partitions}\\ \min\{j,k\}\leq -2}} G_{\mu,\xi,j,k}(q, z) \cdot \lambda_1^j \lambda_2^k c_\mu(V)c_\xi(E)$$

for some series $G_{\mu,\xi,j,k} \in \mathbb{Q}[\![q, z]\!]$.

Our goal is to get a universal series expression by exponentiating the above equality. To do so, we first show the terms in the second summation vanish using a similar approach as in Section 3.3. This proves that $\mathcal{N}_S(E, V; q, z)$ is admissible, from which we deduce the desired expressions by taking the limits of Lemma 3.7.

Let S' be a toric projective surface with a natural action by $\mathsf{T}_0 = (\mathbb{C}^*)^2$. Say the fixed points are $p_1, \dots, p_M$ and the Chern roots of the tangent space of S' at p_i are $a_1^{(i)}, a_2^{(i)}$, which live in $H^*_{\mathsf{T}_0}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \lambda_2]$. Let E', V' be two arbitrary T-equivariant bundles on S' with Chern roots $b_1^{(i)}, \dots, b_N^{(i)}$ and $c_1^{(i)}, \dots, c_r^{(i)}$ respectively at p_i. By a virtual version of the argument in Section 3.3, this time via the virtual Bott residue formula, we have

$$\mathcal{N}_{S'}(E', V'; q, z) = \left(\prod_{i=1}^M \mathcal{N}_S(E, V; q, z)\Big|_{\vec{\lambda}\rightsquigarrow\vec{a}^{(i)},\vec{m}\rightsquigarrow\vec{b}^{(i)},\vec{w}\rightsquigarrow\vec{c}^{(i)}}\right)\Bigg|_{\vec{\lambda}=\vec{m}=\vec{w}=0}$$

where the symbol $\rightsquigarrow$ denotes a substitution of variables. After the substitution inside the bracket of the right hand side, we know from equivariant integration that the resulting expression is a power series in $\vec{\lambda}, \vec{m}, \vec{w}$, which is why we are able to set these variables to 0 and obtain a number. Now we focus on the bundles

$$E' = \bigoplus_{j=1}^{N} \mathcal{O}_{S'}\langle y_j \rangle, \quad V' = \bigoplus_{j=1}^{r} \mathcal{O}_{S'}\langle v_j \rangle$$

whose Chern roots at each p_i are $m_1, \dots, m_N$ and $w_1, \dots, w_r$ respectively, independent of i. Thus

$$\mathcal{N}_{S'}(E', V'; q, z) = \left(\prod_{i=1}^{M} \mathcal{N}_S(E, V; q, z)|_{\vec{\lambda} \rightsquigarrow \vec{a}^{(i)}} \right) \Bigg|_{\vec{\lambda}=\vec{w}=\vec{m}=0}.$$

Again, note that the term inside the bracket is a power series with respect to λ_1, λ_2. Substituting the previous expansion of $\log \mathcal{N}_S(E, V; q, z)$, we see that

$$\begin{aligned} \log \mathcal{N}_{S'}(E', V'; q, z) = &\left(\sum_{i=1}^{M} \sum_{\substack{\mu,\xi \text{ partitions} \\ \min\{i,j\} \geq -1}} G_{\mu,\xi,i,j}(q, z) \cdot \lambda_1^j \lambda_2^k c_\mu(V) c_\xi(E) \Bigg|_{\vec{\lambda} \rightsquigarrow \vec{a}^{(i)}} \right) \Bigg|_{\vec{\lambda}=\vec{w}=\vec{m}=0} \\ &+ \left(\sum_{i=1}^{M} \sum_{\substack{\mu,\xi \text{ partitions} \\ \min\{i,j\} \leq -2}} G_{\mu,\xi,i,j}(q, z) \cdot \lambda_1^j \lambda_2^k c_\mu(V) c_\xi(E) \Bigg|_{\vec{\lambda} \rightsquigarrow \vec{a}^{(i)}} \right) \Bigg|_{\vec{\lambda}=\vec{w}=\vec{m}=0}. \end{aligned}$$

Since the elementary symmetric polynomials form a basis for symmetric polynomials, we know the coefficients in front of q, z of the term

$$\sum_{i=1}^{M} \sum_{j,k \in \mathbb{Z}} G_{\mu,\xi,j,k}(q, z) \cdot \lambda_1^j \lambda_2^k \Bigg|_{\vec{\lambda} \rightsquigarrow \vec{a}^{(i)}} \tag{3.13}$$

are power series in λ_1, λ_2 for each μ, ξ.

Let $S' = \mathbb{P}^1 \times \mathbb{P}^1$, with T_0-action

$$(t_1, t_2) \cdot ([x_0 : x_1], [y_0 : y_1]) = ([x_0 : t_1 x_1], [y_0, t_2 y_1]).$$

We refer to [LYZ02, Section 3.7] for the following computations of equivariant weights. The fixed points are

$$p_1 = p_{00} = ([1 : 0], [1 : 0]), \quad p_2 = p_{01} = ([1 : 0], [0 : 1]),$$
$$p_3 = p_{10} = ([0 : 1], [1 : 0]), \quad p_4 = p_{11} = ([0 : 1], [0 : 1]).$$

The corresponding weights are $\vec{a}^{(1)} = \vec{a}^{(00)}, \vec{a}^{(2)} = \vec{a}^{(01)}, \vec{a}^{(3)} = \vec{a}^{(10)}, \vec{a}^{(4)} = \vec{a}^{(11)}$ where

$$a_1^{(ij)} = (-1)^i \lambda_1, \quad a_2^{(ij)} = (-1)^j \lambda_2$$

for $i, j \in \{0, 1\}$. Substituting into (3.13), we see the summands with odd j or k would cancel each other out, leaving us with

$$\sum_{j,k \text{ even}} G_{\mu,\xi,j,k}(q, z) \cdot 4\lambda_1^j \lambda_2^k.$$

Since $\lambda_1^j \lambda_2^k$ are linearly independent for all distinct j, k, and they are not polynomials for $\min\{j, k\} \leq -2$, the coefficients $G_{\mu,\xi,j,k}$ must all be 0 for these j and k.

Having dealt with the case where j, k are both even, we would like to apply the same argument to the other cases. To do so we need to solve the problem that the summands vanish whenever one of j, k is odd. The fixed points on $\mathrm{Quot}_{S'}(E', n)$ correspond to M-tuples of N-coloured partitions $(\mu^{(1)}, \dots, \mu^{(M)})$, where $\mu^{(i)} = (\mu^{(i,1)}, \dots, \mu_i^{(i,N)})$ and each $\mu^{(i,j)}$ is a partition for $i = 1, \dots, M$ and

$j=1,\dots,N$. If we replace w_k by $u_k(l+\lambda_1+\lambda_2)^2$ for some symmetric polynomial p and numbers $l,u_1,\dots,u_r$, the Chern roots of $(\mathcal{O}_{S'}\langle w_k\rangle)^{[n]}$ would be replaced by

$$\bigcup_{i=1}^{M}\bigcup_{j=1}^{N}\bigcup_{\square\in\mu^{(i,j)}}\{u_k\cdot(l+a_1^{(i)}+a_2^{(i)})^2+m_j-c(\square)a_1^{(i)}-r(\square)a_2^{(i)}\}.$$

We claim that taking symmetric series of these Chern roots would result in terms composed of symmetric series of Chern roots of $K_{S'}^{[n]}$, and that of $\mathcal{O}_{S'}^{[n]}$, which are respectively given by the sets

$$\bigcup_{i=1}^{M}\bigcup_{j=1}^{N}\bigcup_{\square\in\mu^{(i,j)}}\{a_1^{(i)}+a_2^{(i)}+m_j-c(\square)a_1^{(i)}-r(\square)a_2^{(i)}\},$$

$$\bigcup_{i=1}^{M}\bigcup_{j=1}^{N}\bigcup_{\square\in\mu^{(i,j)}}\{m_j-c(\square)a_1^{(i)}-r(\square)a_2^{(i)}\}.$$

This is a result of Lemma 3.10 by setting $\vec{x}=(a_1^{(i)}+a_2^{(i)})_{i=1,\dots M}$, $\vec{y}=(m_j-c(\square)a_1^{(i)}-r(\square)a_2^{(i)})_{i=1,\dots,M}^{\square\in\mu^{(i,j)}}$. Therefore after replacing w_k by $u_k(l+a_1^{(i)}+a_2^{(i)})$, the resulting invariant is still an integral of characteristic classes of tautological bundles, so the new version of (3.13) remains a power series in λ_1,λ_2. View $u_1,\dots,u_r$ as formal variables and replace them with $w_1,\dots,w_r$, we see in total we have replaced each w_k by $(l+a_1^{(i)}+a_2^{(i)})w_k$ for $k=1,\dots,r$.

As a result of the previous paragraph, the coefficients of

$$\begin{aligned}&\sum_{i=1}^{M}\sum_{j,k}G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(l+\lambda_1+\lambda_2)^{2|\mu|}\Big|_{\vec{\lambda}\rightsquigarrow\vec{a}^{(i)}}\\ =&\sum_{s=0}^{2|\mu|}\sum_{i=1}^{M}\sum_{j,k}\binom{2|\mu|}{s}G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot l^{2|\mu|-s}(\lambda_1+\lambda_2)^s\Big|_{\vec{\lambda}\rightsquigarrow\vec{a}^{(i)}}\end{aligned}$$

are power series in λ_1,λ_2 for any integer $l\geq 0$. When $\mu\neq(0)$, the matrix formed by the vectors

$$\left(\binom{2|\mu|}{0}l^{2|\mu|},\binom{2|\mu|}{1}l^{2|\mu|-1},\binom{2|\mu|}{2}l^{2|\mu|-2},\dots,\binom{2|\mu|}{2|\mu|}l^0\right)$$

for $l=1,2,3,\dots$ has maximal rank, we may take a linear combination of the above expression, and get that

$$\sum_{i=1}^{M}\sum_{j,k\in\mathbb{Z}}G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(\lambda_1+\lambda_2)^s\Big|_{\vec{\lambda}\rightsquigarrow\vec{a}^{(i)}}$$

is a power series in λ_1,λ_2 for each $s=0,1,\dots,2|\mu|$.

Take $s=2$, we get

$$\begin{aligned}&\sum_{i=1}^{M}\sum_{j,k}G_{\mu,\xi,j,k}(q,z)\cdot\lambda_1^j\lambda_2^k\cdot(\lambda_1+\lambda_2)^2\Big|_{\vec{\lambda}\rightsquigarrow\vec{a}^{(i)}}\\ =&\sum_{j,k\text{ odd}}G_{\mu,\xi,j,k}(q,z)\cdot 8\lambda_1^{j+1}\lambda_2^{k+1}.\end{aligned}$$

Again, since $\lambda_1^{j+1}\lambda_2^{k+1}$ are linearly independent for distinct j,k and are not polynomials in λ_1,λ_2 for any $\min\{j,k\}\leq -2$, we know $G_{\mu,\xi,j,k}=0$ whenever j,k are both odd and $\mu\neq(0)$.

In the case one of j, k is odd and the other is even, continuing to assume $\mu \neq (0)$, we take $s = 1$ and get

$$\sum_{i=1}^{M} G_{\mu,\xi,j,k}(q,z) \cdot \lambda_1^j \lambda_2^k \cdot (\lambda_1 + \lambda_2)\Big|_{\vec{\lambda} \rightsquigarrow \vec{a}^{(i)}}$$
$$= \begin{cases} G_{\mu,\xi,j,k}(q,z) \cdot 8\lambda_1^{j+1}\lambda_2^k, \text{ if } j \text{ odd, } k \text{ even} \\ G_{\mu,\xi,j,k}(q,z) \cdot 8\lambda_1^{j}\lambda_2^{k+1}, \text{ if } k \text{ odd, } j \text{ even.} \end{cases}$$

Although these are not polynomials when $\min\{j,k\} \leq -2$, we see there might be some linear dependence, i.e. we could have

$$G_{\mu,\xi,j,k} = -G_{\mu,\xi,j+1,k-1}$$

for j odd and k even, and terms canceling each other out in the sum. To solve this issue, we further apply the argument to the $s = 3$ case and obtain the following dependencies

$$G_{\mu,\xi,j,k} = -G_{\mu,\xi,j+3,k-3}$$

for all j odd, k even and $\min\{j,k\} \leq -4$. Combining these relations we see for $\min\{j,k\} \leq -2$, there exist some constants $C^{\pm}_{\mu,\xi,a,b,l}$, labeled by the partitions μ, ξ, integers a, b, l and a sign $\pm$, such that

$$G_{\mu,\xi,j,k}(q,z) = \sum_{a,b}((-1)^j - (-1)^k)C^{\pm}_{\mu,\xi,a,b,j+k}q^a z^b$$
$$= \begin{cases} \sum_{a,b} 2C^{+}_{\mu,\xi,a,b,j+k}q^a z^b & \text{if } j \text{ even, } k \text{ odd, } j \geq 0 \\ \sum_{a,b} -2C^{+}_{\mu,\xi,a,b,j+k}q^a z^b & \text{if } j \text{ odd, } k \text{ even, } j > 0, \\ \sum_{a,b} 2C^{-}_{\mu,\xi,a,b,j+k}q^a z^b & \text{if } j \text{ even, } k \text{ odd, } j < 0 \\ \sum_{a,b} -2C^{-}_{\mu,\xi,a,b,j+k}q^a z^b & \text{if } j \text{ odd, } k \text{ even, } j < 0, \end{cases}$$

The reason for the superscript $\pm$ is due to cases such as $j = -1, k = 0$, where we would have $\min\{j,k\} > -2$, so the dependence does not necessarily hold. Because of this gap, we can not always relate the coefficient when $j \geq 0$ to $j < 0$, resulting in separated cases. By the paragraphs preceding this theorem, for a fixed a, the degrees j, k on λ_1, λ_2 of the $[q^a]$ coefficient are bounded below. However the above indicates that the constants $C^{\pm}_{\mu,\xi,a,b,l}$ only depend on the value $l = j + k$, and we can make j or k arbitrarily small. Hence $C^{\pm}_{\mu,\xi,a,b,l} = 0$ whenever $\mu \neq (0)$.

With all the vanishings of $G_{\mu,\xi,j,k}$, we write

$$\log \mathcal{N}_S(E,V;q,z) = \sum_{\substack{\mu,\xi \text{ partitions} \\ j,k \geq -1}} G_{\mu,\xi,j,k}(q,z) \cdot \lambda_1^j \lambda_2^k \cdot c_\mu(V)$$
$$+ \sum_{\substack{\xi \text{ partition} \\ \min\{j,k\} \leq -2}} G_{(0),\xi,j,k}(q,z) \cdot \lambda_1^j \lambda_2^k.$$

To deal with the terms $G_{(0),\xi,j,k}(q,z)$ for $\min\{j,k\} \leq -2$, we apply Lemma 4.1 and find

$$D_z G_{(0),\xi,j,k}(q,z) = kG_{(1),\xi,j,k}(q,z) = 0,$$

so $G_{(0),j,k}$ is constant with respect to the variable z. Let us attempt to extract the $[q^n \lambda_1^j \lambda_2^k c_{(0)}(V) c_\xi(E)]$ coefficient of the Chern series from $G_{(0),\xi,j,k}$ using the Chern limit of Lemma 3.7. This results in a limit $\varepsilon \to 0$ of the term $\varepsilon^{n(N-r)}\varepsilon^{|\xi|}\varepsilon^{j+k}$, which does not make sense when the rank r is sufficiently large, so we must have $G_{(0),\xi,j,k} = 0$ for such r. To generalize this to arbitrary ranks, we apply

[GM22, Lemma 3.3] to $\mathcal{N}_S$, which says the coefficients of $\mathcal{N}_S$ are polynomials in r when $r \geq 0$. Now we can write

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{\substack{\mu,\xi \text{ partitions} \\ j,k \geq -1}} G_{\mu,\xi,j,k}(q, z) \cdot \lambda_1^j \lambda_2^k \cdot c_\mu(V) c_\xi(E).$$

As noted in [OP22, Equation (31)], the obstruction on $\mathrm{Hilb}^n(S)$ at a fixed point $[Z_\mu]$ is $(K_S^{[n]})^\vee|_{Z_\mu}$. From (3.2), we see a copy of $K_S^\vee = t_1 t_2$ is in $K_S^{[n]}|_{Z_\mu}$. By (3.9), the obstruction bundle on $\mathrm{Quot}_S(E, n)$ at any fixed-point has at least one copy of $K_S^\vee$ as a direct summand. For a line bundle L, we have

$$\mathrm{ch}(\Lambda_{-1} L^\vee) = 1 - e^{-c_1(L)} = e(L) \cdot (1 + \dots)$$

where $\dots$ are some omitted terms in $H_{\mathsf{T}}^{\geq 0}(\mathrm{pt})$. Therefore $1/\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{vir}})^\vee)$ has a factor of $e(K_S^\vee) = c_1(S) = \lambda_1 + \lambda_2$ in its numerator. We also note that this factor does not appear in the denominator because if we pass to the subtorus $\{(t_1, t_2) : t_1 t_2 = 1\}$, the Zariski tangent space has no T-fixed parts: by (3.7), the fixed part can only come from the direct summands with $i = j$, which correspond to the Hilbert scheme case; but by (3.3), these summands have no fixed parts because $a(\Box), l(\Box) \geq 0$ for any box $\Box$. Therefore we may extract this factor of $c_1(S)$ and obtain

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{\substack{\mu,\xi \text{ partitions} \\ j,k \geq -1}} H_{\mu,\xi,j,k}(q, z) \cdot \lambda_1^j \lambda_2^k \cdot c_\mu(V) c_\xi(E) c_1(S).$$

for some series $G_{\mu,\xi,j,k} \in \mathbb{Q}[\![q, z]\!]$. Furthermore, since j, k are now bounded below by -1, multiplying by $\lambda_1 \lambda_2$ would give us a power series expansion in λ_1, λ_2, allowing us to use the symmetry in λ_1, λ_2 and write

$$\log \mathcal{N}_S(E, V; q, z) = \sum_{\mu,\nu,\xi \text{ partitions}} H_{\mu,\nu,\xi}(q, z) \cdot \int_S c_\mu(V) c_\nu(S) c_\xi(E) c_1(S). \tag{3.14}$$

for some series $H_{\mu,\nu,\xi} \in \mathbb{Q}[\![q, z]\!]$.

Finally, taking Chern and Verlinde limits of $H_{\mu,\nu,\xi}$ as in Lemma 3.7, then exponentiating gives us series $C_{\mu,\nu,\xi}, B_{\mu,\nu,\xi}$ such that

$$\mathcal{C}_S(E, V; q) = \prod_{\mu,\nu,\xi \text{ partitions}} C_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V) c_\nu(S) c_\xi(E) c_1(S)},$$

$$\mathcal{V}_S(E, V; q) = \prod_{\mu,\nu,\xi \text{ partitions}} B_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V) c_\nu(S) c_\xi(E) c_1(S)}.$$

and the fact that $\mathcal{S}_S(E, V; q) = \mathcal{C}_S(E, -V; q)$ implies that there exists series $A_{\mu,\nu,\xi}$ such that

$$\mathcal{S}_S(E, V; q) = \prod_{\mu,\nu,\xi \text{ partitions}} A_{\mu,\nu,\xi}(q)^{\int_S c_\mu(V) c_\nu(S) c_\xi(E) c_1(S)}.$$

To generalize this to arbitrary K-theory classes $\alpha = [V'] - [V''] \in K_{\mathsf{T}}(S)$ for equivariant bundles V', V'' of rank m, l respectively, we apply [GM22, Lemma 3.3] once more; it states that the invariants for α are obtained by substituting

$$r \rightsquigarrow m - l, \quad p_n(v_1, v_2, \dots, v_r) \rightsquigarrow p_n(v_1', v_2', \dots, v_m') - p_n(v_1'', v_2'', \dots, v_l''),$$

where p_n are the power-sum symmetric polynomials. Hence the above universal series expressions hold for all $\alpha \in K_{\mathsf{T}}(S)$. $\Box$

Lemma 3.10. *Let $F(\vec{x}, \vec{y})$ be a polynomial symmetric in $\vec{x} = (x_1, \dots, x_n)$ and symmetric in $\vec{y} = (y_1, \dots, y_m)$, then F can be written as a polynomial expression of symmetric functions in $\{y_1, \dots, y_m\}$ and symmetric functions in $\{x_i + y_j\}_{i=1,\dots,n}^{j=1,\dots,m}$.*

Proof. Expand F in the elementary symmetric polynomial basis with respect to the variables $y_1, \dots, y_m$:

$$F(\vec{x}, \vec{y}) = \sum_{\mu \text{ partition}} f_\mu(\vec{x}) e_\mu(\vec{y}).$$

An induction on the degree of F shows that $f_\mu(\vec{x})$ can be written in the desired form for $\mu \neq (0)$. Thus we may assume F is independent of y. Furthermore, since if the statement holds for F and G, then it holds for $F + G$ and $F \cdot G$, we can assume $F = e_k(\vec{x})$. We have

$$e_k(\{x_i + y_j\}_{i=1,\dots,n}^{j=1,\dots,m}) = K \cdot e_k(\vec{x}) + G(\vec{x}, \vec{y})$$

for some constant $K \in \mathbb{Z}$, and G is a polynomial symmetric in $\vec{x}$ and in $\vec{y}$, and every monomial term in G contains some y_j. Apply the same argument to G and we conclude that G satisfies the claim, therefore so does F by the above equation. □

By the non-equivariant Segre-Verlinde correspondence [Boj21a, Theorem 1.7] and the relations between the non-equivariant series and equivariant series illustrated in Section 3.3, we have a weak Segre-Verlinde correspondence as the following corollary. The same argument of the following proof also gives us a weak symmetry in the form of Corollary 1.12.

Corollary 3.11. *In the setting of Theorem 3.9, we have the following correspondence*

$$A_{\mu,\nu,\xi}(q) = B_{\mu,\nu,\xi}((-1)^N q).$$

whenever one of μ, ν, ξ *is* (1) *and the other two are* (0)*. In particular, the degree 0 part satisfies*

$$\mathcal{S}_{S,0}(E, \alpha; q) - \mathcal{V}_{S,0}(E, \alpha; -q) = \sum_{n=2}^{\infty} \frac{f_n}{(\lambda_1 \lambda_2)^{n-2}} \cdot \left(\int_S c_1(S) \right)^2 \cdot q^n$$

for some terms $f_n \in H_{\mathsf{T}}^{2n-2}(pt)$ *dependent on* α.

Proof. By the argument of Section 3.3, the universal series in Theorem 3.9, when passed to a toric projective surface, must give the Segre-Verlinde correspondence of [Boj21a, Theorem 1.7] in degree 0. Since the degree 0 terms occur only when one of μ, ν, ξ is (1) and the other two are (0), we have

$$A_{\mu,\nu,\xi}(q) = B_{\mu,\nu,\xi}((-1)^N q)$$

in those cases.

Note that when we take exp of (3.14), the total degree 0 part might come from the product of a negative-degree term and a positive-degree term, but since each term in the integrand is accompanied by a copy $c_1(S)$, we know this difference must be a multiple of $c_1(S)^2$. We also see the $[q^n]$ coefficients are sums of products of at most n such integrals, giving a denominator of $\lambda_1^n \lambda_2^n$, so we are done.

For illustration, we shall extract this difference, and express it explicitly. This is just a standard computation. For a partition $\mu = (\mu_1, \mu_2, \dots, \mu_L)$ of size n with length L, and a sequence of positive integers $\kappa = (k_1, k_2, \dots, k_L)$, write $\kappa | \mu$ if each $k_i | \mu_i$. We also associate a set of tuples of partitions to κ by

$$M_\kappa := \left\{ \left((\mu^{(i)})_{i=1}^L, (\nu^{(i)})_{i=1}^L, (\xi^{(i)})_{i=1}^L \right) \;\middle|\; \begin{array}{l} \mu^{(i)}, \nu^{(i)}, \xi^{(i)} \text{ are partitions for each } i, \text{ s.t.} \\ \quad \sum_{i=1}^L k_i(|\mu^{(i)}| + |\nu^{(i)}| + |\xi^{(i)}| - 1) = 0 \text{ and} \\ \mu_i = \nu_i = \xi_i = 0 \text{ for some } i. \end{array} \right\}.$$

For each $n > 0$, we would like to find the degree 0 part of the $[q^n]$ coefficient of exp of (3.14). By expanding the exponential using definition, we observe that these terms come from products of integrals labeled by $\mu^{(i)}, \nu^{(i)}, \xi^{(i)}$ in M_κ for some tuples $\kappa | \pi$ for some partition π of size n. A more precise description is given by the following equation. Suppose $\log(A_{\mu,\nu,\xi}(q)) = \sum_{i=1}^{\infty} a_{\mu,\nu,\xi,i}q^i$ and $\log(B_{\mu,\nu,\xi}(q)) = \sum_{i=1}^{\infty} b_{\mu,\nu,\xi,i}q^i$, we have

$$\begin{aligned}
&[q^n](\mathcal{S}_{S,0}(E,\alpha;q) - \mathcal{V}_{S,0}(E,\alpha;(-1)^N q)) \\
&= \sum_{\substack{|\pi|=n \\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec{\mu},\vec{\nu},\vec{\xi})\in M_\kappa} \prod_{i=1}^{\ell(\pi)} \frac{1}{k_i!} \left(a_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i} \int_S c_{\mu^{(i)}}(\alpha) c_{\nu^{(i)}}(S) c_{\xi^{(i)}}(E) c_1(S) \right)^{k_i} \\
&\quad - (-1)^{Nn} \sum_{\substack{|\pi|=n \\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec{\mu},\vec{\nu},\vec{\xi})\in M_\kappa} \prod_{i=1}^{\ell(\pi)} \frac{1}{k_i!} \left(b_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i} \int_S c_{\mu^{(i)}}(\alpha) c_{\nu^{(i)}}(S) c_{\xi^{(i)}}(E) c_1(S) \right)^{k_i} \\
&= \sum_{\substack{|\pi|=n \\ \ell(\pi)>1}} \sum_{\kappa|\pi} \prod_{(\vec{\mu},\vec{\nu},\vec{\xi})\in M_\kappa} \left(\prod_{i=1}^{\ell(\pi)} a^{k_i|M_k|}_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i} - (-1)^{Nn} \prod_{i=1}^{\ell(\pi)} b^{k_i|M_k|}_{\mu^{(i)},\nu^{(i)},\xi^{(i)},\pi_i/k_i} \right) \\
&\quad \cdot \prod_{i=1}^{\ell(\pi)} \frac{c_1(S)^{k_i}}{k_i! \lambda_1^{k_i} \lambda_2^{k_i}} \left(c_{\mu^{(i)}}(\alpha) c_{\nu^{(i)}}(S) c_{\xi^{(i)}}(E) \right)^{k_i}.
\end{aligned}$$

In the summation we have $\ell(\pi) > 1$ because M_κ is empty for any $\kappa|\pi$ if $\pi = (n)$ by definition. Therefore $2 \leq \sum_{i=1}^{\ell(\pi)} k_i \leq n$. By multiplying some appropriate power of $\lambda_1\lambda_2$ to the denominator and numerator of the right hand side, we can express it as a rational function in λ_1, λ_2, with denominator $\lambda_1^n \lambda_2^n$ and numerator a multiple of $c_1(S)^2$. Setting this multiple as $f_n \in \mathbb{C}[\lambda_1, \lambda_2]$ gives the desired expression.

□

Example 3.12. The universal series for $\mathcal{N}_S$ are known explicitly in the compact case [Boj21a, Theorem 1.2]. For a smooth projective surface S and α of rank r, we apply [AJL$^+$21, Theorem 17, Equations (16),(17)] for $f(x) = 1 - ze^x, g(x) = \frac{x}{1-e^{-x}}$ and get

$$\mathcal{N}_S(\mathcal{O}_S, \alpha; q, z) = \left[\left(\frac{1-zQ}{1-z} \right)^r \left(\frac{-rzQ(1-Q)}{1-zQ} + 1 \right) \right]^{c_1(S)^2} \left[\frac{1-zQ}{1-z} \right]^{c_1(S)\cdot c_1(\alpha)}$$

via the substitution

$$q = \frac{1-Q^{-1}}{(1-zQ)^r}.$$

As mentioned in the introduction, when $N = 1$, we shall set $y_1 = 1$, and omit the subscript ξ for $H_{\mu,\nu,\xi}$. The formula above allows us to compute $H_{(1),(0)}(q,z)$ and $H_{(0),(1)}(q,z)$. For a smooth projective surface S and α of rank 0, we have

$$\mathcal{N}_S(\mathcal{O}_S, \alpha, q, z) = \left(\frac{1-q-z}{(1-q)(1-z)} \right)^{c_1(S)\cdot c_1(\alpha)}.$$

The exponent is interpreted as intersection product, which in the toric case corresponds to the equivariant push-forward

$$\int_S c_1(\alpha) c_1(S).$$

Therefore for rank 0, we have the following series from expansion (3.14)

$$H_{(1),(0)}(q,z) = \log \frac{1-q-z}{(1-q)(1-z)}, \quad H_{(0),(1)}(q,z) = 0,$$

Take the Chern limit of Lemma 3.9 by substituting $q \rightsquigarrow -q\varepsilon$, $z \rightsquigarrow (1+\varepsilon)^{-1}$ and get

$$C_{(1),(0)}(q) = 1 + q$$

Replacing α by $-\alpha$ in the Chern series to get the Segre series for α, we see

$$A_{(1),(0)}(q) = C_{(1),(0)}(q)^{-1} = \frac{1}{1+q}$$

On the other hand, the Verlinde limit yields

$$B_{(1),(0)}(q) = \frac{1}{1-q}$$

The Segre-Verlinde correspondence of Corollary 3.11 is indeed satisfied.

Example 3.13. Let $S = \mathbb{C}^2$, $n = N = 2$, $E = \mathcal{O}_S\langle y_1\rangle \oplus \mathcal{O}_S\langle y_2\rangle$ and $L = \mathcal{O}_S\langle v\rangle$. The T_1-fixed locus of $\mathrm{Quot}_S(E, n)$ is the disjoint union of

$$\mathrm{Hilb}^0(S) \times \mathrm{Hilb}^2(S), \quad \mathrm{Hilb}^1(S) \times \mathrm{Hilb}^1(S), \quad \mathrm{Hilb}^2(S) \times \mathrm{Hilb}^0(S)$$

Denote Z_μ the point in $\mathrm{Hilb}^i(S)^{\mathsf{T}_0}$ corresponding to a partition μ, then the T-fixed points of $\mathrm{Quot}_S(\mathbb{C}^2, 2)$ are

$$(Z_\phi, Z_{(2)}), (Z_\phi, Z_{(1,1)}), (Z_{(1)}, Z_{(1)}), (Z_{(2)}, Z_\phi), (Z_{(1,1)}, Z_\phi)$$

Therefore by (3.7), the virtual tangent bundles at these five points are respectively

$$(t_1^2 + t_2 - t_1^2 t_2 + y_1^{-1} y_2)(1 + t_1^{-1})$$
$$(t_2^2 + t_1 - t_1 t_2^2 + y_1^{-1} y_2)(1 + t_2^{-1})$$
$$(t_1 + t_2 - t_1 t_2)(1 + y_1^{-1} y_2)(1 + y_1 y_2^{-1})$$
$$(t_1^2 + t_2 - t_1^2 t_2 + y_1 y_2^{-1})(1 + t_1^{-1})$$
$$(t_2^2 + t_1 - t_1 t_2^2 + y_1 y_2^{-1})(1 + t_2^{-1})$$

The equivariant Chern roots of $\alpha^{[n]}$ at these points are respectively

$\{m_2+w, m_2-\lambda_1+w\}, \{m_2+w, m_2-\lambda_2+w\}, \{m_1+w, m_2+w\}, \{m_1+w, m_1-\lambda_1+w\}, \{m_1+w, m_1-\lambda_2+w\}$

The contribution to the Segre numbers at each of these fixed points are

$$\frac{(2\lambda_1+\lambda_1)(\lambda_1+\lambda_2)}{2(m_1-m_2+\lambda_1)(m_1-m_2)(m_2-\lambda_1+w_1+1)(m_2+w_1+1)(\lambda_2-\lambda_1)\lambda_1^2\lambda_2},$$
$$\frac{(\lambda_1+2\lambda_2)(\lambda_1+\lambda_2)}{2(m_1-m_2+\lambda_2)(m_1-m_2)(m_2-\lambda_2+w_1+1)(m_2+w_1+1)(\lambda_1-\lambda_2)\lambda_1\lambda_2^2},$$
$$\frac{(\lambda_1+\lambda_2+m_1-m_2)(\lambda_1+\lambda_2-m_1+m_2)(\lambda_1+\lambda_2)^2}{(m_1-m_2+\lambda_1)(m_1-m_2-\lambda_1)(m_1-m_2+\lambda_2)(m_1-m_2-\lambda_2)(m_1+w1+1)(m_2+w1+1)\lambda_1^2\lambda_2^2},$$
$$\frac{(2\lambda_1+\lambda_1)(\lambda_1+\lambda_2)}{2(m_1-m_2-\lambda_1)(m_1-m_2)(m_1-\lambda_1+w_1+1)(m_1+w_1+1)(\lambda_2-\lambda_1)\lambda_1^2\lambda_2},$$
$$\frac{(\lambda_1+2\lambda_2)(\lambda_1+\lambda_2)}{2(m_1-m_2-\lambda_2)(m_1-m_2)(m_1-\lambda_2+w_1+1)(m_1+w_1+1)(\lambda_1-\lambda_2)\lambda_1\lambda_2^2}.$$

Summing them up, we have

$$[q^2]S^2(\alpha;q) = \frac{\begin{pmatrix} m_1m_2\lambda_1 - m_1\lambda_1^2 - m_2\lambda_1^2 + \lambda_1^3 + m_1m_2\lambda_2 - 3m_1\lambda_1\lambda_2 - 3m_2\lambda_1\lambda_2 + 3\lambda_1^2\lambda_2 - m_1\lambda_2^2 - m_2\lambda_2^2 \\ +3\lambda_1\lambda_2^2 + \lambda_2^3 + m_1\lambda_1 w + m_2\lambda_1 w - 2\lambda_1^2 w + m_1\lambda_2 w + m_2\lambda_2 w - 6\lambda_1\lambda_2 w - 2\lambda_2^2 w \\ +\lambda_1 w^2 + \lambda_2 w^2 + m_1\lambda_1 + m_2\lambda_1 - 2\lambda_1^2 + m_1\lambda_2 + m_2\lambda_2 - 6\lambda_1\lambda_2 - 2\lambda_2^2 + 2\lambda_1 w + 2\lambda_2 w + \lambda_1 + \lambda_2 \end{pmatrix}(\lambda_1+\lambda_2)}{2(m_1-\lambda_1+w+1)(m_1-\lambda_2+w+1)(m_1+w+1)(m_2-\lambda_1+w+1)(m_2-\lambda_2+w+1)(m_2+w+1)\lambda_1^2\lambda_2^2}.$$

A similar computation yields another complicated expression for the Verlinde number. We are interested in the total degree 0 part of their difference in the variables $\vec{\lambda}, \vec{m}, \vec{w}$, this computes to

$$\begin{aligned}[q^2](S_{S,0}(E,L;q) - V_{S,0}(E,L;q)) &= -\frac{(3m_1m_2 - \lambda_1\lambda_2 + 3m_1w + 3m_2w + 3w^2)(\lambda_1+\lambda_2)^2}{3\lambda_1^2\lambda_2^2} \\ &= \left(\frac{1}{3}c_2(S) - c_2(E) - c_1(E)c_1(V) - c_1(V)^2\right)\left(\int_S c_1(S)\right)^2.\end{aligned}$$

Note that even though the expressions for Segre and Verlinde numbers are complicated, their difference in degree 0 simplifies tremendously and satisfies Corollary 3.11.

3.6. **Reduced virtual classes and invariants.** As mentioned previously, the obstruction for $\mathrm{Quot}_S(E,n)$ at Z contains at least one copy of $K_S^\vee$. For K-trivial surfaces, this causes the Euler class of T^{vir} to vanish. Therefore the virtual Verlinde and Segre numbers both vanish. One can instead study the "reduced" versions of these invariants. By [Lim21, Proposition 9], when S is a K-trivial surface, $n > 0$, and E a torsion free sheaf, there is a *reduced obstruction theory* that is perfect in the sense of Definition 3.5. The reduced (virtual) tangent bundle in this case is given by adding a trivial summand to the usual virtual tangent bundle:

$$T^{\mathrm{red}} = T^{\mathrm{vir}} + \mathcal{O}_{\mathrm{Quot}_S(E,n)}.$$

In this section, we study the equivariant analogue where $S = \mathbb{C}^2$ with the natural action of the the 1-dimensional torus

$$\mathsf{T}_0 = \mathbb{C}^* = \{(t_1,t_2) : t_1t_2 = 1\}.$$

Write

$$H^*_{\mathsf{T}_0}(\mathrm{pt}) = \mathbb{C}[\lambda_1,\lambda_2]/(\lambda_1+\lambda_2) = \mathbb{C}[\lambda],$$
$$K_{\mathsf{T}_0}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}]/(t_1t_2-1) = \mathbb{Z}[t^{\pm1}].$$

Using the argument of [CK17, Lemma 3.1], we see that the $\mathsf{T} = (\mathsf{T}_0 \times \mathsf{T}_1 \times \mathsf{T}_2)$-fixed locus of $\mathrm{Quot}_S(E,n)$ stays unchanged, and the Zariski tangent space at each of the fixed points has no fixed parts by the descriptions (3.3) and (3.7). The *equivariant reduced Segre and Verlinde series* $\mathcal{S}_S^{\mathrm{red}}$ and $\mathcal{V}_S^{\mathrm{red}}$ are defined the same as the virtual ones by replacing T^{vir} with T^{red}. Here we omit the subscript since we are only interested in the $S = \mathbb{C}^2$ case.

$$\mathcal{S}^{\mathrm{red}}(E,\alpha;q) := \sum_{n>0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{c(\alpha^{[n]}|_Z)}{e(T_Z^{\mathrm{red}})},$$
$$\mathcal{V}^{\mathrm{red}}(E,\alpha;q) := \sum_{n>0}^{\infty} q^n \sum_{Z \in \mathrm{Quot}_S(E,n)^{\mathsf{T}}} \frac{\mathrm{ch}(\det(\alpha^{[n]}|_Z))}{\mathrm{ch}(\Lambda_{-1}(T_Z^{\mathrm{red}})^\vee)}.$$

Note that we do not include the $n = 0$ term because condition 2 of [Lim21, Proposition 9] is only satisfied when $n > 0$.

The same strategy from the previous section can be applied to study these invariants. For $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $V = \oplus_{i=1}^r \mathcal{O}_S\langle v_i\rangle$, define

$$\mathcal{N}^{\mathrm{red}}(E,V;q,z) := \sum_{\mu \neq (0)} q^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{j=1}^N \prod_{i=1}^r (1 - t^{-c(\square)+r(\square)} v_i y_j z)}{\mathrm{ch}(\Lambda_{-1}(T^{\mathrm{red}}|_Z)^\vee)}.$$

Again note that the $[q^0]$ coefficient is 0. We can think of the reduced obstruction as removing a copy of $K_S^\vee$ from the usual obstruction in $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}]$, then passing to the quotient ring $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}]/(t_1t_2-1)$. This gives us the following result.

Corollary 3.14. *For $n > 0$, the $[q^n]$ coefficient of $\mathcal{N}^{red}$ can be obtained from the non-reduced version by taking the following limit:*

$$[q^n]\mathcal{N}^{red}(E,V;q,z) = [q^n]\frac{\mathcal{N}_S(E,V;q,z)}{1-e^{-c_1(K_S^\vee)}}\bigg|_{-\lambda_2\to\lambda_1=\lambda}$$
$$= [q^n]\lim_{-\lambda_2\to\lambda_1=\lambda}\frac{\mathcal{N}_S(E,V;q,z)}{\lambda_1+\lambda_2}.$$

Expand the universal series expression from Theorem 3.9 and obtain

$$\mathcal{S}_S(E,\alpha;q) = \sum_{i=1}^{\infty}\frac{1}{i!}(\lambda_1+\lambda_2)^i\left(\sum_{\mu,\nu,\xi}\log A_{\mu,\nu,\xi}(q,z)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)\right)^i.$$

Using the above corollary to extract the reduced coefficients, we see the terms with $i > 1$ all vanish and

$$\mathcal{S}^{\rm red}(E,\alpha;q) = \sum_{\mu,\nu,\xi}\log A_{\mu,\nu,\xi}(q,z)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E).$$

The Chern and Verlinde cases are similar, thus we have the following result.

Theorem 3.15. *When $S = \mathbb{C}^2$, the equivariant reduced Segre and Verlinde series for $E = \oplus_{i=1}^N \mathcal{O}_S\langle y_i\rangle$ and $\alpha \in K_{\mathsf{T}}(S)$ are*

$$\mathcal{S}^{red}(E,\alpha;q) = \sum_{\mu,\nu,\xi}\log\left(A_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{V}^{red}(E,\alpha;q) = \sum_{\mu,\nu,\xi}\log\left(B_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E),$$
$$\mathcal{C}^{red}(E,\alpha;q) = \sum_{\mu,\nu,\xi}\log\left(C_{\mu,\nu,\xi}(q)\right)\cdot\int_S c_\mu(\alpha)c_\nu(S)c_\xi(E)$$

where $A_{\mu,\nu,\xi}, B_{\mu,\nu,\xi}$ and $C_{\mu,\nu,\xi}$ are the same series from Theorem 1.7.

The integrals in the above theorem labeled by μ,ν,ξ have degree $|\mu|+|\nu|+|\xi|-2$, which is one degree lower than the integrals in the non-reduced expressions. Therefore we have a Segre-Verlinde correspondence in degree -1 for the reduced setting. However, results for degree -1 have no compact analogues since they automatically vanish in the compact setting. In the Section 4.3, we compute some of the universal series explicitly, giving us some Segre-Verlinde relations in non-negative degrees for the reduced case.

4. Explicit computations of universal series on $\mathrm{Hilb}^n(\mathbb{C}^2)$

4.1. **Equivariant Segre number at rank $r = -2$.** We shall compute the explicit expression for the equivariant (non-virtual) Chern series at rank $r = 2$ for the Hilbert schemes. First recall the Chern limit from Proposition 3.4,

$$I^{\mathcal{C}}(V;q) = \lim_{\varepsilon\to 0}\Omega\left(-q\varepsilon^{2-r}(1+\varepsilon)^r;\frac{e^{-\varepsilon w_1}}{1+\varepsilon},\ldots,\frac{e^{-\varepsilon w_r}}{1+\varepsilon};e^{\varepsilon\lambda_1},e^{\varepsilon\lambda_2}\right). \tag{4.1}$$

By definition, the Chern number $[q^n]I^{\mathcal{C}}(V;q)$ of any rank 2 equivariant bundle V has total degree $-2n$ to 0 in λ_1,λ_2. Thus by comparing coefficients, we see the positive degree terms in (3.5) must vanish after taking this limit. The following lemma, based on a similar statement mentioned in [GM22, Section 4.6], allows us to compute the remaining negative degree coefficients

$$H_{(0),-1,-1}(q,z), H_{(1),-1,-1}(q,z), H_{(0),-1,0}(q,z)$$

from the degree 0 coefficients given by (3.6).

Lemma 4.1. *Let $r \geq 0$ and suppose $H(z_1, \dots, z_r)$ is a power series in $z_1, \dots z_r$ whose coefficients are series in some other variables $q_1, q_2, \dots$. If H is symmetric in $z_1, \dots, z_r$ with expansion*

$$H(ze^{x_1}, \dots, ze^{x_r}) = \sum_{\mu \text{ partition}} H_\mu(z) \prod_{i=1}^{\ell(\mu)} e_{\mu_i}(x_1, \dots, x_r),$$

then for any $k \geq 0$, we have

$$D_z^k H_{(0)}(z) = k! \sum_{|\mu|=k} \binom{r}{\mu} H_\mu(z)$$

where $\binom{r}{\mu}$ denotes $\prod_{i=1}^{\ell(\mu)} \binom{r}{\mu_i}$, and $D_z = z\frac{\partial}{\partial z}$.

Proof. We begin by claiming that the statement is closed under polynomial expressions; that is, if the equality holds for both $F(z_1, \dots, z_r)$ and $G(z_1, \dots, z_r)$, then it holds for $F \cdot G$ and $aF + G$ for any $a \in \mathbb{Q}[\![q_1, q_2, \dots]\!]$. The additive part is straightforward, and we shall prove the multiplicative part of this claim. Expand

$$F(ze^{x_1}, \dots, ze^{x_r}) = \sum_{\mu \text{ partition}} F_\mu(z) e_\mu(x_1, \dots, x_r),$$
$$G(ze^{x_1}, \dots, ze^{x_r}) = \sum_{\mu \text{ partition}} G_\mu(z) e_\mu(x_1, \dots, x_r),$$

then $H = F \cdot G$ can be expanded as

$$H(ze^{x_1}, \dots, ze^{x_r}) = \sum_{\mu \text{ partition}} H_\mu(z) e_\mu(x_1, \dots, x_r) = \sum_{\nu+\xi=\mu} F_\nu(z) G_\xi(z) e_\mu(x_1, \dots, x_r)$$

where by $\nu + \xi$ we mean combining them as sequences to get a partition of size $|\nu| + |\xi|$ with length $\ell(\nu) + \ell(\xi)$. Suppose the statement holds for both F and G, then we have

$$\begin{aligned} D_z^k H_{(0)}(z) &= D_z^k \left(F_{(0)}(z) G_{(0)}(z)\right) \\ &= \sum_{i=1}^{k} \binom{k}{i} D_z^i F_{(0)} D_z^{k-i} G_{(0)} \\ &= \sum_{i=1}^{k} \binom{k}{i} i!(k-i)! \sum_{|\nu|=i, |\xi|=k-i} \binom{r}{\nu} F_\nu \binom{r}{\xi} G_\xi \\ &= k! \sum_{|\mu|=k} \binom{r}{\mu} H_\mu. \end{aligned}$$

Since $H(z_1, \dots, z_r)$ is symmetric, by the above observation, it suffices to prove the statement when H is the power sum symmetric polynomial $p_n(z_1, \dots, z_r) = z_1^n + \cdots + z_r^n$. For each $n \geq 0$, we expand

$$H(ze^{x_1}, \dots, ze^{x_r}) = p_n(ze^{x_1}, \dots, ze^{x_r}) = \sum_{j=1}^{r} z^n e^{nx_j} = z^n \left(r + \sum_{i>0} \frac{n^i}{i!} p_i(x_1, \dots, x_r) \right).$$

This means $H_{(0)}(z) = rz^n$ and

$$D_z^k H_{(0)}(z) = rn^k z^n.$$

Fix $r \geq 1$, we write

$$p_n(x_1, \dots, x_r) = \sum_{|\mu|=n} C_\mu e_\mu(x_1, \dots, x_r)$$

for some constant terms C_μ. Evaluate at $x_1 = \cdots = x_r = 1$ and get

$$r = \sum_{|\mu|=n} \binom{r}{\mu} C_\mu.$$

Hence

$$k! \sum_{|\mu|=k} \binom{r}{\mu} H_\mu(z) = z^n n^k \sum_{|\mu|=k} \binom{r}{\mu} C_\mu = r n^k z^n = D_z^k H_{(0)}(z).$$

A quick calculation for the $k = 0$ or $r = 0$ cases finishes the proof.

□

We would like to apply the above lemma to the series from (3.6). First, using the admissibility of Ω combined with an expansion as used in (2.3), we obtain

$$\log \Omega(q; z_1, \dots, z_r; e^{-\lambda_1}, e^{-\lambda_2}) = \sum_{k_1, k_2 \geq -1} G_{k_1,k_2}(q; z_1, \dots, z_r) \int_S E_{k_1,k_2}(c_1(S), c_2(S))$$

for some series $G_{k_1,k_2}(q, z_1, \dots, z_r)$. Comparing to (3.5), we see

$$G_{k_1,k_2}(q; ze^{w_1}, \dots, ze^{w_r}) = \sum_{\mu \text{ partition}} (-1)^{k_1+k_2} H_{\mu,k_1,k_2}(q, z) c_\mu(V)$$

Apply Lemma 4.1, where we set $H(z_1, \dots, z_r)$ in the lemma to be $G_{k_1,k_2}(q; z_1, \dots, z_r)$ and the variables $x_1, \dots, x_r$ to be $w_1, \dots w_r$, we conclude

$$D_z^k H_{(0),k_1,k_2}(q, z) = k! \sum_{|\mu|=k} \binom{r}{\mu} H_{\mu,k_1,k_2}(q, z).$$

When $r = 2$, the series $H_{(2),-1,-1}, H_{(1,1),-1,-1}$ and $H_{(1),-1,0}$ are known explicitly by (3.6). This allows us to calculate $H_{(1),-1,-1}, H_{(0),-1,-1}$ and $H_{(0),-1,0}$ by taking anti-derivatives using the identities

$$\mathrm{Li}_1 = -\log(1-q), \quad D_z \operatorname{Li}_n(qz^k) = k \operatorname{Li}_{n-1}(qz^k)$$

for $k, n > 0$. They are obtained as follows

$$\begin{aligned} H_{(1),-1,-1}(q, z) &= -\operatorname{Li}_2(qz^2) + \operatorname{Li}_2(qz), \\ H_{(0),-1,-1}(q, z) &= h_{(0),-1,-1}(q) - \operatorname{Li}_3(qz^2) + 2\operatorname{Li}_3(qz), \\ H_{(0),-1,0}(q, z) &= h_{(0),-1,0}(q) + \frac{1}{2}\operatorname{Li}_2(qz^2) - \operatorname{Li}_2(qz), \end{aligned} \tag{4.2}$$

for some terms $h_{(0),-1,-1}(q), h_{(0),-1,0}(q)$ independent of z. By (3.5) and the following lemma, we have

$$\begin{aligned} h_{(0),-1,-1}(q) &= [\lambda^{-2}] \log \Omega(q; 0, \dots, 0; e^\lambda, e^\lambda) = -\operatorname{Li}_3(q), \\ h_{(0),-1,0}(q) &= [\lambda^{-1}] \frac{1}{2} \log \Omega(q; 0, \dots, 0; e^\lambda, e^\lambda) = \frac{1}{2} \operatorname{Li}_2(q) \end{aligned}$$

where the factor $\frac{1}{2}$ in the second line is due to $H_{(0),-1,0} = H_{(0),0,-1}$.

Lemma 4.2. *The following identity is satisfied:*

$$\log \Omega(Q; 0, \dots, 0; e^\lambda, e^\lambda) = -\sum_{n=1}^\infty \frac{1}{n} \frac{Q^n}{(1-e^{n\lambda})^2}.$$

Proof. By [GM22, Theorem 2.3],

$$\begin{aligned}&\Omega(Q; z_1, \dots, z_r; q_1, q_2)\\ =&\operatorname{Exp}\left(-\frac{Q+\sum z_i}{(1-q_1)(1-q_2)}\right)\cdot\sum_{\mu}(-1)^{|\mu|}\frac{\tilde{H}_\mu[Q+1;q_1,q_2]\tilde{H}_\mu[z_1+\dots+z_r;q_1,q_2]T_\mu(q_1,q_2)}{N_\mu(q_1,q_2)}.\end{aligned}$$

Here $T_\mu(q_1,q_2)=\prod_{\square\in\mu}q_1^{c(\square)}q_2^{r(\square)}$, $N_\mu(q_1,q_2)=\prod_{\square\in\mu}(q_1^{a(\square)+1}-q_2^{l(\square)})(q_1^{a(\square)}-q_2^{l(\square)+1})$, and $\tilde{H}_\mu$ is the *modified Macdonald polynomial* defined by [GH93, Theorem 1]. Evaluating at $z_1=\dots=z_n=0$, the term $\tilde{H}_\mu[z_1+\dots+z_r;q_1,q_2]$ vanishes unless $\mu=(0)$, and when $\mu=(0)$, we have $\tilde{H}_\mu=1$. This can be seen from its Schur function expansion [GH93, Definition 1]

$$\tilde{H}_\mu[X;q_1,q_2]=\sum_{|\nu|=|\mu|}s_\nu[X]\tilde{K}_{\mu\nu}(q_1,q_2)$$

where $\tilde{K}_{\mu\nu}$ are the modified Macdonald-Kostka polynomials. Therefore when $z_1=\dots=z_r=0$, the summation in the above equation evaluates to 1, and

$$\Omega(Q;0,\dots,0;q_1,q_2)=\operatorname{Exp}\left(-\frac{Q}{(1-q_1)(1-q_2)}\right)$$

By definition of plethystic exponential,

$$\log\Omega(Q;0,\dots,0;q_1,q_2)=-\sum_{n=1}^{\infty}\frac{1}{n}\frac{Q^n}{(1-q_1^n)(1-q_2^n)}.$$

Substituting $q_1=q_2=e^{\lambda}$ gives the desired form. □

Now we can compute the Chern series. In order to take the limit (4.1) of $H_{\mu,k_1,k_2}(q,z)$, we need to substitute $q\rightsquigarrow -q\varepsilon^{2-r}(1+\varepsilon)^r$ and $z\rightsquigarrow(1+\varepsilon)^{-1}$. In (3.5), the series H_{μ,k_1,k_2} is multiplied by a homogeneous function of degree k_1+k_2 in λ_1,λ_2, as well as a homogeneous function of degree $|\mu|$ in $w_1,\dots,w_r$. The limit requires us to substitute $\vec{\lambda}\rightsquigarrow-\varepsilon\vec{\lambda}$ and $\vec{w}\rightsquigarrow-\varepsilon\vec{w}$. Therefore before taking the limit $\varepsilon\to 0$, we need to multiply by $(-\varepsilon)^{|\mu|+k_1+k_2}$. Applying this procedure to (3.6) and (4.2), the limit of $H_{(0),-1,-1}, H_{(1),-1,-1}, H_{(2),-1,-1}$ in the $r=2$ case all return $\log(1+q)$, while the other terms all vanish. Hence (4.1) yields

$$\begin{aligned}I^{\mathcal{C}}(V;q)=&\exp\left(\log(1+q)\int_S c_0(V)+\log(1+q)\int_S c_1(V)+\log(1+q)\int_S c_2(V)\right)\\ =&(1+q)^{\int_S c(V)}\end{aligned}\tag{4.3}$$

for any T-equivariant bundle V of rank 2.

4.2. **Virtual Segre number in rank $r=-1$.** Recall that on Hilbert schemes, the obstruction theory at a fixed point $[Z_\mu]$ is given by $(K_S^{[n]})^\vee|_{Z_\mu}$, so

$$\frac{1}{e(T^{\mathrm{vir}}_{Z_\mu})}=\frac{e((K_S^{[n]})^\vee|_{Z_\mu})}{e(T_{Z_\mu})}=(-1)^{|\mu|}\frac{e((K_S^{[n]})|_{Z_\mu})}{e(T_{Z_\mu})}.$$

Let $L=\mathcal{O}_S\langle v_1\rangle$ be an equivariant line bundle, and $V=L\oplus\mathcal{O}_S\langle v_2\rangle$. Apply (4.3) to V and we have

$$I^{\mathcal{C}}(V;q)=(1+q)^{\int_S c(V)}=(1+q)^{\int_S(1+w_1+w_2+w_1w_2)}.$$

Set $w_2=c_1(K_S)-1$ and replace q by $-q$, then this becomes

$$I^{\mathcal{C}}(V;-q)|_{w_2=c_1(K_S)-1}=(1-q)^{\int_S c(L)c_1(K_S)}.$$

On the other hand, we have by definition

$$
\begin{aligned}
&I^{\mathcal{C}}(V;-q)|_{w_2=c_1(K_S)-1}\\
&=\sum_{\mu}(-1)^{|\mu|}q^{|\mu|}\prod_{\square\in\mu}\frac{(1+w_1-c(\square)\lambda_1-r(\square)\lambda_2)(c_1(K_S)-c(\square)\lambda_1-r(\square)\lambda_2)}{((a(\square)+1)\lambda_1-l(\square)\lambda_2)((l(\square)+1)\lambda_2-a(\square)\lambda_1)}\\
&=\sum_{\mu}(-1)^{|\mu|}q^{|\mu|}\frac{c(L^{[n]}|_{Z_\mu})e((K_S^{[n]})|_{Z_\mu})}{e(T|_{Z_\mu})}\\
&=\sum_{\mu}q^{|\mu|}\frac{c(L^{[n]}|_{Z_\mu})}{e(T^{\mathrm{vir}}|_{Z_\mu})}=\mathcal{C}_S(\mathcal{O}_S,L;q)
\end{aligned}
$$

Therefore we conclude

$$\mathcal{C}_S(\mathcal{O}_S,L;w)=(1-q)^{\int_S c(L)c_1(K_S)}=\left(\frac{1}{1-q}\right)^{\int_S c(L)c_1(S)}.$$

In particular, restricting to the lowest degree part in the variables λ_1,λ_2,w_1, we obtain the following Corollary.

Corollary 4.3. *For $S=\mathbb{C}^2$, the following equality holds*

$$\sum_{n=1}^{\infty}q^n\int_{[\mathrm{Hilb}^n(S)]^{vir}}1:=\sum_{Z\in\mathrm{Hilb}^n(S)^{\mathsf{T}}}\frac{1}{e(T_Z^{vir})}=e^{\frac{(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}q}.$$

4.3. **Segre-Verlinde correspondence in positive degrees.** Recall that we use the notation $H_{\mu,\nu,\xi}$ for the series from (3.14) describing the virtual Nekrasov genus for Quot schemes on $\mathbb{C}^2$. For a smooth projective surface S, a torsion free sheaf E of rank N, and a K-theory class α of rank r, we apply [Boj21a, Equation (4.1), Theorem 4.1] by setting $f(x)=1-ze^x$, $g(x)=\frac{x}{1-e^{-x}}$ and get

$$\mathcal{N}_S(E,V;q,z)=\left(\prod_{i=1}^N F(H_i)\right)^{c_1(S)c_1(\alpha)}\left(\prod_{i=1}^N F(H_i)\right)^{\frac{r}{N}c_1(S)c_1(E)}G(R)^{c_1(S)^2}.$$

Here $R=f^rg^N$, $F(x)=\frac{f(x)}{f(0)}$, the series $H_i(q)$ are Newton–Puiseux solutions to

$$H_i^N=qR(H_i),$$

and $G(R)$ is given by [Boj21a, Equation (4.24)]. Therefore

$$
\begin{aligned}
H_{(1),(0),(0)}(q,z)&=\sum_{i=1}^N\log F(H_i)\\
H_{(0),(1),(0)}(q,z)&=\log G(R)\\
H_{(0),(0),(1)}(q,z)&=\frac{r}{N}\sum_{i=1}^N\log F(H_i)
\end{aligned}
\tag{4.4}
$$

Now let $S=\mathbb{C}^2$. Note that $\mathcal{N}_S$ satisfies

$$
\begin{aligned}
\mathcal{N}_S(y_1,\dots,y_N;v_1,\dots,v_r;q,z)&=\mathcal{N}_S(y_1,\dots,y_N;ze^{w_1},\dots,ze^{w_r};q,1)\\
&=\mathcal{N}_S(ze^{m_1},\dots,ze^{m_N};v_1,\dots,v_r;q,1).
\end{aligned}
$$

Applying Lemma 4.1 to $H_{\mu,\nu,\xi}$ in the variables $w_1,\dots,w_r$ and gives us for $r>0$,

$$D_z^kH_{(0),\nu,\xi}(q,z)=rD_z^{k-1}H_{(1),\nu,\xi}(q,z)=k!\sum_{|\mu|=k}\binom{r}{\mu}H_{\mu,(0)}(q,z),$$

while applying in the variables $m_1, \dots, m_N$ yields

$$D_z^k H_{\mu,\nu,(0)}(q,z) = N D_z^{k-1} H_{\mu,\nu,(1)}(q,z) = k! \sum_{|\xi|=k} \binom{N}{\mu} H_{\mu,\nu,\xi}(q,z). \tag{4.5}$$

When the rank r is negative, we consider $\alpha = -[V]$ where $V = \oplus_{i=1}^{-r} \mathcal{O}_S \langle v_i \rangle$, and the same argument applies. Thus for all $r \neq 0$,

$$D_z^k H_{(0),\nu,\xi}(q,z) = |r| D_z^{k-1} H_{(1),\nu,\xi}(q,z) = k! \sum_{|\mu|=k} \binom{|r|}{\mu} H_{\mu,\nu,\xi}(q,z). \tag{4.6}$$

Our goal for this section is to apply Chern and Verlinde limits to (4.5) and (4.6), together with the explicit expressions (4.4), and obtain relations between the Chern and Verlinde series for various μ, ν and ξ.

4.3.1. *The Chern limit.* For an arbitrary series $f(q,z)$, we define its *Chern limit of degree k and rank r* by

$$\lim_{\varepsilon \to 0} (-\varepsilon)^k f\left((-1)^N q \varepsilon^{N-r} (1+\varepsilon)^r, \frac{1}{1+\varepsilon}\right).$$

Note that when applied to $H_{\mu,\nu,\xi}$, the Chern limit of degree $k = |\mu| + |\nu| + |\xi| - 1$ in this sense agrees with the Chern limit of Lemma 3.7. These limits can be computed using the following analogue of [GM22, Lemma 4.2].

Lemma 4.4. *Suppose $f(q,z) = \sum_{m,n \geq 0} f_{m,n} q^m z^n$ such that*

$$f_{m,n} = (-1)^n p_m(n) \binom{rm}{n}$$

for some polynomial $p_m(x)$ of degree at most $mN + k$. The Chern limit of degree k for f is

$$\lim_{\varepsilon \to 0} (-\varepsilon)^k f\left((-1)^N q \varepsilon^{N-r} (1+\varepsilon)^r, \frac{1}{1+\varepsilon}\right) = \sum_{m=0}^{\infty} [x^{mN+k}] p_m(x) (rm)_{(mN+k)} q^m.$$

Proof. First observe that both sides of the identity are polynomials in r, so it suffices to prove the equality for r large enough. We shall assume $r > \max\{N+k, 0\}$, then $\binom{rm}{n}$ vanishes when $n > rm$. Let $g(q,\varepsilon) = f\left((-1)^N q \varepsilon^{N-r}(1+\varepsilon)^r, (1+\varepsilon)^{-1}\right)$, $f_m(z) = \sum_{n=0}^{rm} f_{m,n} z^n$ then

$$[q^m] g(q,\varepsilon) = (-1)^{mN} f_m\left(\frac{1}{1+\varepsilon}\right) (1+\varepsilon)^{rm} \varepsilon^{mN-rm} = \sum_{i=0}^{rm} c_i \varepsilon^{mN-i}$$

for some numbers c_i. Substitute $\varepsilon = z^{-1} - 1$ we have

$$\begin{aligned}
\sum_{n=0}^{rm} (-1)^n p_m(n) \binom{rm}{n} z^n =& f_m(z) \\
=& (-1)^{mN} \sum_{i=0}^{rm} c_i z^i (1-z)^{rm-i} \\
=& (-1)^{mN} \sum_{i=0}^{rm} c_i \sum_{n=i}^{rm} (-1)^{n-i} \binom{rm-i}{n-i} z^n.
\end{aligned}$$

Therefore

$$p_m(x) = \sum_{i=0}^{rm} (-1)^{mN+i} \frac{(x)_{(i)}}{(rm)_{(i)}} c_i.$$

Since the degree of $p_m(x)$ must be at most $mN+k$, we can extract the coefficient

$$[x^{mN+k}]p_m(x) = (-1)^k \frac{c_{m+k-1}}{(rm)_{(mN+k)}}$$

and get

$$\begin{aligned}\lim_{\varepsilon\to 0}(-\varepsilon)^k f\left((-1)^N q\varepsilon^{N-r}(1+\varepsilon)^r, \frac{1}{1+\varepsilon}\right) &= \sum_{m=0}^{\infty}(-1)^k c_{mN+k}q^m \\ &= \sum_{m=0}^{\infty}[x^{mN+k}]p_m(x)(rm)_{(mN+k)}q^m.\end{aligned}$$

□

4.3.2. *The Verlinde limit.* For an arbitrary series $f(q,z)$, its *Verlinde limit of rank* r is

$$\lim_{\varepsilon\to 0} f((-1)^r q\varepsilon^r, \varepsilon^{-1}).$$

Suppose $r>0$ and $f(q,z)\in\mathbb{Q}[z][\![q]\!]$. Write $f(q,z)=\sum_{m=0}^{\infty} f_m(z)$, then this limit makes sense when f_m are polynomials of degree at most rm. In this case, we have for each $m>0$,

$$[q^m]\lim_{\varepsilon\to 0} f((-1)^r q\varepsilon^r, \varepsilon^{-1}) = (-1)^{rm}[z^{rm}]f_m(z)$$

Now suppose $r<0$ and $f\in\mathbb{Q}[\![q,z]\!]$. For each $m>0$, the limit extracts the following coefficients term-wise

$$[q^m]\lim_{\varepsilon\to 0} f((-1)^r q\varepsilon^r, \varepsilon^{-1}) = (-1)^{rm}[z^{-rm}]f_m(z^{-1}).$$

4.3.3. *Computations.* We begin with the case $\nu=(0),\xi=(0)$ and apply the Chern limit of Lemma 3.7 to $|r|D_z^{k-1}H_{(1),(0),(0)}(q,z)$ for $k>0$. By [Boj22, Corollary 2], we have

$$[q^m]H_{(1),(0),(0)} = \frac{1}{m}[t^{mN-1}]\left(-e^t z(1-e^t z)^{rm-1}\left(\frac{t}{1-e^{-t}}\right)^{mN}\right).$$

We may extract the z^n coefficient and get

$$\begin{aligned}[z^n q^m]H_{(1),(0),(0)} &= \frac{(-1)^n}{m}\binom{rm-1}{n-1}[t^{mN-1}]\left(e^{tn}\left(\frac{t}{1-e^{-t}}\right)^{mN}\right) \\ &= (-1)^n\frac{n}{rm^2}[t^{mN-1}]\left(e^{tn}\left(\frac{t}{1-e^{-t}}\right)^{mN}\right)\binom{rm}{n}.\end{aligned}$$

By the identity $D_z(z^n)=nz^n$, we have

$$|r|[z^n q^m]D^{k-1}H_{(1),(0),(0)} = (-1)^n\frac{n^k|r|}{rm^2}[t^{mN-1}]\left(e^{tn}\left(\frac{t}{1-e^{-t}}\right)^{mN}\right)\binom{rm}{n}$$

Since the right hand side of (4.6) consists of universal series whose powers have degree $k-1$, we take the Chern limit of degree $k-1$ via the Lemma 4.4 by setting $p_m(x)=\frac{x^k}{rm^2}[t^{mN-1}](e^{tx}(\frac{t}{1-e^{-t}})^{mN})$.

Then (4.6) gives us

$$[q^m]k!\sum_{|\mu|=k}\binom{|r|}{\mu}\log C_{\mu,(0),(0)}(q,z) = \frac{|r|(rm)_{(mN+k-1)}}{rm^2}[x^{mN-1}t^{mN-1}]\left(e^{tx}\left(\frac{t}{1-e^{-t}}\right)^{mN}\right)$$

$$= \frac{|r|(rm)_{(mN+k-1)}}{rm^2(mN-1)!}$$

$$= \frac{|r|}{m}\binom{mr-1}{mN-1}(m(r-N))_{(k-1)},$$

$$[q^m]\sum_{|\mu|=k}\binom{|r|}{\mu}\log C_{\mu,(0),(0)}(q,z) = \frac{|r|}{mk}\binom{mr-1}{mN-1}\binom{m(r-N)}{k-1}.$$

Observe that the Verlinde series for $\alpha \in K_{\mathsf{T}}(S)$ only depends on $c_1(\alpha)$ by definition. Therefore the universal series are non-trivial only when $\mu = (1)_k := (1,\ldots,1)$ has k copies of 1. Taking the Verlinde limit, we get

$$k!|r|^k \log B_{(1)_k,(0),(0)}(q,z) = (-1)^{rm}|r|(|r|m)^{k-1}[z^{|r|m}q^m]H_{(1),(0),(0)}$$

$$= \frac{|r|(|r|m)^{k-1}}{m}[t^{mN-1}]\left(\frac{e^{rmt}t^{mN}}{(1-e^{-t})^{mN}}\right)$$

$$= |r|^k m^{k-2}\operatorname{res}_{t=0}\frac{e^{rmt}}{(1-e^{-t})^{mN}}$$

where res refers to the residue of a meromorphic function. Let $0<R<1$. We compute the above residue by integrating along the rectangular contour formed by the points $\{\pm R \pm i\pi\}$:

$$2\pi i \operatorname{res}_{t=0}\frac{e^{rmt}}{(1-e^{-t})^{mN}} = \int_{-R}^{R}\frac{e^{rm(z+i\pi)}}{(1-e^{-(z+i\pi)})^{mN}}dz - \int_{-R}^{R}\frac{e^{rm(z-i\pi)}}{(1-e^{-(z-i\pi)})^{mN}}dz$$

$$+ i\int_{-\pi}^{\pi}\frac{e^{rm(R+iz)}}{(1-e^{-(R+iz)})^{mN}}dz - i\int_{-\pi}^{\pi}\frac{e^{rm(-R+iz)}}{(1-e^{-(-R+iz)})^{mN}}dz$$

The first two integrals cancel each other out because $e^{i\pi} = e^{-i\pi}$. The third integral gives us

$$i\int_{-\pi}^{\pi}\frac{e^{rm(R+i\theta)}}{(R-e^{-(1+i\theta)})^{mN}}d\theta = \oint \frac{z^{rm-1}}{(1-z^{-1})^{mN}}dz$$

$$= 2\pi i \operatorname{res}_{z=0} z^{mN+rm-1}(z-1)^{-mN}$$

$$= 2\pi i\binom{m(r+N)-1}{mr}.$$

Similarly, we could show the last integral vanishes. Hence for $r \neq 0$, we have

$$[q^m]\log B_{(1)_k,(0),(0)}(q) = \frac{m^{k-2}}{k!}\binom{m(r+N)-1}{mr}.$$

Note that $H_{(0),(0),(1)} = \frac{r}{N}H_{(1),(0),(0)}$. Thus a similar argument using (4.5) gives us

$$[q^m]\sum_{|\xi|=k}\binom{N}{\xi}\log C_{(0),(0),\xi}(q) = \frac{r}{mk}\binom{mr-1}{mN-1}\binom{m(r-N)}{k-1},$$

$$[q^m]\sum_{|\xi|=k}\binom{N}{\xi}\log B_{(0),(0),\xi}(q) = \frac{|r|^{k-1}Nm^{k-2}}{k!}\binom{m(r+N)-1}{mN}.$$

Alternatively, when $r > 0$, we can combine (4.5) and (4.6) and obtain for $k_1, k_2 \geq 1$,

$$[q^m] \sum_{|\mu|=k_1} \sum_{\xi=k_2} \binom{r}{\mu}\binom{N}{\xi} \log C_{\mu,(0),\xi}(q) = \frac{r(r-N)}{k_1 k_2} \binom{mr-1}{mN-1} \binom{m(r-N)-1}{k_1-1, k_2-1},$$

$$[q^m] \sum_{\xi=k_2} \binom{N}{\xi} \log B_{(1)_{k_1},(0),\xi}(q) = \frac{r^{k_2} m^{k_1+k_2-2}}{k_1! k_2!} \binom{m(r+N)-1}{mr}.$$

This yields the following Segre-Verlinde relations for terms with $\nu = (0)$.

Theorem 4.5. *For rank $r \neq 0$ and $n, k > 0$, the universal series of Theorem 3.9 satisfy the following identities*

$$\begin{aligned}
&[q^m] \sum_{|\mu|=k} \binom{|r|}{\mu} \log C_{\mu,(0),(0)}(q,z) = \frac{|r|}{mk} \binom{mr-1}{mN-1} \binom{m(r-N)}{k-1},\\
&[q^m] \log B_{(1)_k,(0),(0)}(q) = \frac{m^{k-2}}{k!} \binom{m(r+N)-1}{mr},\\
&[q^m] \sum_{|\xi|=k} \binom{N}{\xi} \log C_{(0),(0),\xi}(q) = \frac{r}{mk} \binom{mr-1}{mN-1} \binom{m(r-N)}{k-1},\\
&[q^m] \sum_{|\xi|=k} \binom{N}{\xi} \log B_{(0),(0),\xi}(q) = \frac{|r|^{k-1} N m^{k-2}}{k!} \binom{m(r+N)-1}{mN}
\end{aligned}$$

where $(1)_k = (1, \ldots, 1)$ is the partition with k copies of 1.

When $r > 0$, we have

$$\begin{aligned}
&[q^m] \sum_{|\mu|=k_1} \sum_{|\xi|=k_2} \binom{r}{\mu}\binom{N}{\xi} \log C_{\mu,(0),\xi}(q) = \frac{r(r-N)}{k_1 k_2} \binom{mr-1}{mN-1} \binom{m(r-N)-1}{k_1-1, k_2-1},\\
&[q^m] \sum_{|\xi|=k_2} \binom{N}{\xi} \log B_{(1)_{k_1},(0),\xi}(q) = \frac{r^{k_2} m^{k_1+k_2-2}}{k_1! k_2!} \binom{m(r+N)-1}{mr}.
\end{aligned}$$

Applying the $k = 2$ case of the above theorem, we get the following Segre-Verlinde correspondence

Corollary 4.6. *The universal series of Theorem 3.9 satisfy the following correspondence*

$$\left(C^{-r,N}_{(1,1),(0),(0)}(q)\right)^{r^2} \left(C^{-r,N}_{(2),(0),(0)}(q)\right)^{\binom{|r|}{2}} = \left(B^{r,N}_{(1,1),(0),(0)}((-1)^N q)\right)^{|r|(r+N)}.$$

$$\left(C^{-r,N}_{(0),(0),(1,1)}(q)^{r^2} C^{-r,N}_{(0),(0),(2)}(q)^{\binom{|r|}{2}}\right)^{|r|} = \left(B^{r,N}_{(0),(0),(1,1)}(-q)^{r^2} B^{r,N}_{(0),(0),(2)}(-q)^{\binom{|r|}{2}}\right)^{r+N}.$$

Remark 4.7. As mentioned in the introduction, combining Theorem 4.5 with Theorem 3.15 yields the corresponding relations for reduced invariants. In particular, Corollary 4.6 implies a correspondence in degree 0 for reduced invariants, which could provide insight into the reduced invariants for K3 surfaces in the compact setting.

5. Segre and Verlinde invariants on $\mathbb{C}^4$

Consider $X = \mathbb{C}^4$ with a $(\mathbb{C}^4)^*$-action by scaling coordinates

$$(t_1, t_2, t_3, t_4) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_2 x_3, t_3 x_3, t_4 x_4).$$

Let $\mathsf{T}_0 = \{(t_1, t_2, t_3, t_4) : t_1t_2t_3t_4 = 1\} \subseteq (\mathbb{C}^4)^*$ be the subtorus which preserves the usual volume form on X, making X a smooth quasi-projective toric Calabi-Yau 4-fold. As in the surface case, we also have the two additional tori

$$\mathsf{T}_1 = (\mathbb{C}^*)^N, \quad \mathsf{T}_2 = (\mathbb{C}^*)^{r+s}$$

where T_1 on $\mathbb{C}^N$, and T_2 acts on $\mathbb{C}^r \times \mathbb{C}^s$, giving us the bundle $E = \oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$ and K-theory class $\alpha = [\oplus_{i=1}^r \mathcal{O}_X\langle v_i\rangle] - [\oplus_{i=r+1}^{r+s} \mathcal{O}_X\langle v_i\rangle]$. Write

$$K_{\mathsf{T}}(\mathrm{pt}) = \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}; y_1^{\pm1}, \dots, y_N^{\pm1}; v_1^{\pm1}, \dots, v_{r+s}^{\pm1}]/(t_1t_2t_3t_4 - 1),$$
$$H^*_{\mathsf{T}}(\mathrm{pt}) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4; m_1, \dots, m_N; w_1, \dots, w_{r+s}]/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4).$$

By [HT08, Theorem 4.1] the truncated Atiyah class of the universal subsheaf $\mathcal{I}$ defines an obstruction theory

$$\mathbf{R}\mathscr{H}om_p(\mathcal{I}, \mathcal{I})_0^\vee[-1] \to L^\bullet_{\mathrm{Quot}_X(E,n)}$$

where $\mathbf{R}\mathscr{H}om_q = \mathbf{R}q_* \circ \mathbf{R}\mathscr{H}om$, $(\cdot)_0$ denotes the trace free part. Note that the obstruction theory is T-equivariant by [Ric21, Theorem B]. The virtual tangent bundle is then

$$T^{\mathrm{vir}} = -\mathbf{R}\mathscr{H}om_p(\mathcal{I}, \mathcal{I})_0 \in K_{\mathsf{T}}(\mathrm{Quot}_X(E, n)).$$

5.1. **Virtual invariants.** When X is a projective Calabi-Yau 4-fold, the virtual fundamental class involves a choice of orientation on $\mathrm{Quot}_X(E, n)$. Let $\mathcal{L} = \det \mathbf{R}\mathscr{H}om_q(\mathcal{I}, \mathcal{I})$ be the determinant line bundle. An *orientation* $o(\mathcal{L})$ is a choice of square root of the isomorphism

$$Q : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_{\mathrm{Quot}_X(E,n)}$$

induced by Serre duality. From this Borisov–Joyce [BJ15] constructed virtual class $[\mathrm{Quot}_X(E, n)]^{\mathrm{vir}}_{o(\mathcal{L})} \in H_{2nN}(\mathrm{Quot}_X(E, n), \mathbb{Z})$. For $\gamma \in H^{2nN}(\mathrm{Quot}_X(E, n))$, the Donaldson-Thomas invariants [CL14] are defined to be

$$\mathrm{DT}_4(\gamma) = \int_{[\mathrm{Quot}_X(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}} \gamma.$$

For the non-compact $X = \mathbb{C}^4$, similar to the surface case, we have that the T-fixed locus of $\mathrm{Hilb}^n(X)$ consists of only finitely many reduced points [CK17, Lemma 3.6], so we can define these invariants equivariantly using Oh-Thomas' localization formula [OT20, Theorem 7.1].

Definition 5.1. For $n > 0$, $\gamma \in H^*_{\mathsf{T}}(\mathrm{Quot}_X(E, n))$, the *Donaldson-Thomas invariants* are

$$\mathrm{DT}_4(X, E, n, \gamma) := \sum_{Z \in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L}|_Z)} \frac{\gamma|_Z}{\sqrt{e}(T^{\mathrm{vir}}|_Z)}$$

where $\sqrt{e}(\cdot)$ is the equivariant version of the Edidin-Graham square root Euler class [EG95a].

Note that this definition depends on a choice of signs at each fixed point Z, which we denote $(-1)^{o(\mathcal{L})|_Z}$. We shall see the virtual tangent bundle at Z is self-dual and admits some square root $\sqrt{T^{\mathrm{vir}}|_Z} \in K_{\mathsf{T}}(\mathrm{pt})$ such that

$$T^{\mathrm{vir}}|_Z = \sqrt{T^{\mathrm{vir}}|_Z} + \overline{\sqrt{T^{\mathrm{vir}}|_Z}},$$

where $\overline{(\cdot)}$ denotes the involution $t_i \mapsto t_i^{-1}$. We then have

$$\sqrt{e}(T^{\mathrm{vir}}|_Z) = \pm e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right),$$

where the sign depends on a choice of orientation, which can be absorbed into the choice of signs $(-1)^{o(\mathcal{L})|_Z}$.

Definition 5.2. Let $X=\mathbb{C}^4$, $\alpha=[\oplus_{i=1}^r\mathcal{O}_X\langle v_i\rangle]-[\oplus_{i=r+1}^{r+s}\mathcal{O}_X\langle v_i\rangle]\in K_{\mathsf{T}}(X)$, and $E=\oplus_{i=1}^N\mathcal{O}_X\langle y_i\rangle$. The *equivariant Segre and Chern series* for a choice of signs $o(\mathcal{L})$ are respectively

$$\begin{aligned}\mathcal{S}_X(E,\alpha;q):=&\sum_{n=0}^{\infty}q^n\sum_{Z\in\mathrm{Quot}_X(E,n)^{\mathsf{T}}}(-1)^{o(\mathcal{L})|_Z}\frac{s(\alpha^{[n]}|_Z)}{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)}\\ &\in\frac{\mathbb{C}(\lambda_1,\lambda_2,\lambda_3,\lambda_4;m_1,\dots,m_N;w_1,\dots,w_{r+s})}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}[q],\\ \mathcal{C}_X(E,\alpha;q):=&\sum_{n=0}^{\infty}q^n\sum_{Z\in\mathrm{Quot}_X(E,n)^{\mathsf{T}}}(-1)^{o(\mathcal{L})|_Z}\frac{c(\alpha^{[n]}|_Z)}{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)}.\end{aligned}$$

To define the Verlinde number, we first consider the *untwisted virtual structure sheaf*

$$\mathcal{O}^{\mathrm{vir}}:=\hat{\mathcal{O}}^{\mathrm{vir}}\otimes\det{}^{\frac{1}{2}}((E^{\vee})^{[n]}). \tag{5.1}$$

where $\hat{\mathcal{O}}^{\mathrm{vir}}$ is the (twisted) virtual structure sheaf of [OT20, Definition 5.9]. For compact X and $\alpha\in K^0(X)$, the *Verlinde series* is defined in [Boj21a, Section 1.3] by

$$\begin{aligned}\mathcal{V}_X(E,\alpha;q):=&\sum_{n=0}^{\infty}q^n\chi^{\mathrm{vir}}\left(\mathrm{Quot}_X(E,n),\det(\alpha^{[n]})\right)\\ :=&\sum_{n=0}^{\infty}q^n\hat{\chi}^{\mathrm{vir}}\left(\mathrm{Quot}_X(E,n),\det(\alpha^{[n]})\otimes\det{}^{\frac{1}{2}}((E^{\vee})^{[n]})\right).\end{aligned}$$

Using the virtual Riemann-Roch formula and equivariant localization of Oh-Thomas [OT20, Theorem 6.1, Theorem 7.3], we have the following equivariant virtual Euler characteristic for $X=\mathbb{C}^4$:

$$\hat{\chi}^{\mathrm{vir}}_{\mathsf{T}}(\mathrm{Quot}_X(E,n),\alpha):=\sum_{Z\in\mathrm{Quot}_X(E,n)^{\mathsf{T}}}(-1)^{o(\mathcal{L})|_Z}e\left(-\sqrt{T^{\mathrm{vir}}|_Z}\right)\sqrt{\mathrm{td}}\left(T^{\mathrm{vir}}|_Z\right)\mathrm{ch}_{\mathsf{T}}(\alpha)$$

where $\sqrt{\mathrm{td}}$ is the the square-root Todd class satisfying

$$\begin{aligned}\sqrt{\mathrm{td}}(T^{\mathrm{vir}}|_Z)=&\,\mathrm{td}\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)\mathrm{ch}\left(\det{}^{\frac{1}{2}}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)\\ =&\frac{e\left(\sqrt{T^{\mathrm{vir}}|_Z}\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)}\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}}^{\frac{1}{2}}\right).\end{aligned}$$

Here we denote

$$K^{\mathrm{vir}}=\det(T^{\mathrm{vir}})^{\vee},\quad \sqrt{K^{\mathrm{vir}}}=\det\sqrt{T^{\mathrm{vir}}}^{\vee}.$$

Substituting into the above equation, we have

$$\hat{\chi}^{\mathrm{vir}}(\mathrm{Quot}_X(E,n),\alpha)=\sum_{Z\in\mathrm{Quot}_X(E,n)^{\mathsf{T}}}(-1)^{o(\mathcal{L})|_Z}\frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}}\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)}\mathrm{ch}(\alpha^{[n]}|_Z).$$

Definition 5.3. The *equivariant Verlinde series* for a choice of signs $o(\mathcal{L})$ is

$$\mathcal{V}_X(E,\alpha;q) := \sum_{n=0}^{\infty} q^n \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}}\right)\mathrm{ch}\left(\det^{\frac{1}{2}}((E^\vee)^{[n]}|_Z)\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee}\right)} \mathrm{ch}\left(\det(\alpha^{[n]}|_Z)\right)$$
$$\in \frac{\mathbb{Q}(t_1,t_2,t_3,t_4;y_1,\dots,y_N;v_1,\dots,v_{r+s})}{(t_1t_2t_3t_4)}[\![q]\!].$$

The relation between Segre and Verlinde numbers in the compact case is studied in [Boj21a] using the *Nekrasov genus* for Hilbert schemes, introduced for the 3-fold case by [NO14]. We consider the following Quot scheme version from [NP19]

$$\mathcal{N}_X(E,\alpha;q) := \sum_{n=0}^{\infty} q^n \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z} \frac{\mathrm{ch}(\sqrt{K^{\mathrm{vir}}|_Z}^{\frac{1}{2}})}{\mathrm{ch}(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_Z}^{\vee})} \mathrm{ch}\left(\frac{\Lambda_{-1}}{\det^{\frac{1}{2}}}\alpha^{[n]}|_Z\right). \tag{5.2}$$

Remark 5.4. Recall in [CKM22], the Nekrasov genus for Hilbert schemes involves a variable y coming from a trivial $\mathbb{C}^*$-action on X. This is exactly the $N=1$ case for the above definition, where we have the parameter y_1 from the T_1-action on E.

5.2. **Vertex formalism.** The invariants in the previous section can be calculated for Hilbert schemes using a vertex formalism developed by [CK20], based on the method introduced in [MNOP06a] for Calabi-Yau 3-folds. We generalize this to Quot schemes using the computations from [Boj21a, Section 2.1]. First when $N=1$, the T-fixed points of $\mathrm{Hilb}^n(X)$ correspond to monomial ideals of $\mathbb{C}[x_1,x_2,x_3,x_4]$ [CK17, Lemma 3.1], which are labeled by solid partitions π of size n where

$$\mathcal{O}_{Z_\pi} = \mathbb{C}[x_1,x_2,x_3,x_4]/I_{Z_\pi} = \mathrm{span}\{x_1^a x_2^b x_3^c x_4^d : (a,b,c,d)\in\pi\}.$$

We denote Q_π the character of $\mathcal{O}_{Z_\pi}$

$$Q_\pi = \sum_{(i,j,k,l)\in\pi} t_1^{-a}t_2^{-b}t_3^{-c}t_4^{-d} \in K^*_{\mathsf{T}}(\mathrm{pt}) = \frac{\mathbb{Z}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]}{(t_1t_2t_3t_4-1)}.$$

Similar to the surface case, for $E=\oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$, the T-fixed points for $\mathrm{Quot}_X(E,n)$ are labeled by N-colored solid partitions $\pi=(\pi^{(1)},\dots,\pi^{(n)})$ of size n, i.e. sequences of form

$$Z_\pi = ([Z_1],[Z_2],\dots,[Z_N]) \in \mathrm{Hilb}^{n_1}(X)\times\cdots\times\mathrm{Hilb}^{n_N}(X)$$

such that each Z_i corresponds by solid partition $\pi^{(i)}$.

Let Q_i be the character of $\mathcal{O}_{Z_i}$. The virtual tangent bundle at Z_π is

$$\begin{aligned} T^{\mathrm{vir}}_{Z_\pi} &= \mathrm{Ext}\left(\bigoplus_{i=1}^N I_{\mathcal{Z}_i}\langle y_i\rangle, \bigoplus_{j=1}^N I_{\mathcal{Z}_j}\langle y_j\rangle\right)_0 \\ &= \sum_{i,j=1}^N \mathcal{O}_X\otimes(1-\overline{P(I_{Z_i})}P(I_{Z_j}))y_i^{-1}y_j \\ &= \sum_{i,j=1}^N \left(Q_j + t_1t_2t_3t_4\overline{Q_i} - t_1t_2t_3t_4P_{1234}\overline{Q_i}Q_j\right) y_i^{-1}y_j \end{aligned} \tag{5.3}$$

where $P(I)$ is the Poincaré polynomial of I defined analogously to (3.8) from previous section, and $P_I=\prod_{i\in I}(1-t_i^{-1})$ for any set of indices I. Specializing $t_1t_2t_3t_4=1$, we get the following (non-unique) square root

$$\sqrt{T^{\mathrm{vir}}_{Z_\pi}} = \sum_{i,j=1}^{N} \left(Q_j - \overline{P_{123}Q_i}Q_j\right) y_i^{-1}y_j.$$

The reason for the above choice of square root is so that

$$\begin{aligned}
\mathrm{ch}\left(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}}\right) &= \mathrm{ch}\left(\prod_{i,j} \det((\overline{(Q_j - \overline{P}_{123}Q_j\overline{Q_i})y_i^{-1}y_j)^{\frac{1}{2}}})\right) \\
&= \mathrm{ch}\left(\prod_{i,j} \det{}^{\frac{1}{2}}(\overline{Q_j})y_iy_j^{-1})\right) \\
&= \frac{1}{\mathrm{ch}\left(\det^{\frac{1}{2}}((E^\vee)^{[n]}|_{Z_\pi})\right)}
\end{aligned}$$

matches our twist in (5.1), and this simplifies our computation as now we have

$$\mathcal{V}_X(E,\alpha;q) = \sum_{\pi}^{\infty} q^{|\pi|}(-1)^{o(\mathcal{L})|_{Z_\pi}} \frac{\mathrm{ch}\left(\det(\alpha^{[n]}|_{Z_\pi})\right)}{\mathrm{ch}\left(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^\vee\right)}$$

The fiber of $V^{[n]} = \oplus_{i=1}^r \mathcal{O}_X^{[n]}\langle v_i\rangle$ over $Z_\pi = (Z_1, \dots Z_N)$ is the rn-dimensional representation

$$V^{[n]}|_{Z_\pi} = \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{N} \mathcal{O}_{Z_j}\langle v_iy_j\rangle = \left(\sum_{i=1}^{r}\sum_{j=1}^{N}\sum_{(a,b,c,d)\in\pi^{(j)}} v_iy_jt_1^{-a}t_2^{-b}t_3^{-c}t_4^{-d}\right) \in K_{\mathsf{T}}(\mathrm{pt}).$$

Therefore for any point Z corresponding to an N-colored solid partition π, we have

$$\begin{aligned}
c(V^{[n]}|_{Z_\pi}) &= \prod_{j=1}^{N}\prod_{(a,b,c,d)\in\pi^{(j)}}\prod_{i=1}^{r}(1 + w_i + m_j - a\lambda_1 - b\lambda_2 - c\lambda_3 - d\lambda_4) \\
\det(V^{[n]}|_{Z_\pi}) &= \prod_{j=1}^{N}\prod_{(a,b,c,d)\in\pi^{(j)}}\prod_{i=1}^{r} v_it_1^{-a}t_2^{-b}t_3^{-c}t_4^{-d}, \\
\mathrm{ch}(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}})\,\mathrm{ch}\left(\frac{\Lambda_{-1}}{\det^{\frac{1}{2}}}V^{[n]}|_{Z_\pi}\right) &= \prod_{j=1}^{N}\prod_{(a,b,c,d)\in\pi^{(j)}} t_1^{\frac{a}{2}}t_2^{\frac{b}{2}}t_3^{\frac{c}{2}}t_4^{\frac{d}{2}} \\
&\quad\cdot\prod_{i=1}^{r}\left(y^{\frac{1}{2}}v_i^{\frac{1}{2}}t_1^{-\frac{a}{2}}t_2^{-\frac{b}{2}}t_3^{-\frac{c}{2}}t_4^{-\frac{d}{2}} - y_j^{-\frac{1}{2}}v_i^{-\frac{1}{2}}t_1^{\frac{a}{2}}t_2^{\frac{b}{2}}t_3^{\frac{c}{2}}t_4^{\frac{d}{2}}\right).
\end{aligned}$$

Using these expressions, we see the Chern and Verlinde series can be extracted by taking limits of the Nekrasov genus, similar to the surface case. Also, it follows that

$$\mathcal{N}_X(E,V;q) \in \frac{\mathbb{Q}(t_1^{\frac{1}{2}},t_2^{\frac{1}{2}},t_3^{\frac{1}{2}},t_4^{\frac{1}{2}})}{(t_1t_2t_3t_4-1)}[\![q, y_1^{\pm\frac{1}{2}},\dots,y_N^{\pm\frac{1}{2}}, v_1^{\pm\frac{1}{2}},\dots,v_r^{\pm\frac{1}{2}}]\!].$$

The argument of [CKM22, Proposition 1.13, 1.15] can be applied to show that $\mathcal{N}_X(E,V;q)$ in fact lives in $\frac{\mathbb{Q}(t_1,t_2,t_3,t_4)}{(t_1t_2t_3t_4-1)}[\![q, y_1^{\pm\frac{1}{2}},\dots,y_N^{\pm\frac{1}{2}}, v_1^{\pm\frac{1}{2}},\dots,v_r^{\pm\frac{1}{2}}]\!]$. This enables us to talk about admissibility (up to specializing $t_1t_2t_3t_4 = 1$) in the sense of Definition 2.6.

Remark 5.5. As n increases, the number of choices for signs $(-1)^{o(\mathcal{L})}$ increases exponentially with the number of fixed points. However, it has been observed in previous studies of the equivariant invariants that statements analogues to the non-equivariant cases hold for some canonical choice of signs [CK17, Nek20]. This choice of signs is conjectured to be unique in for instance [CK17, Conjecture 3.21] and [CKM22, Conjecture 0.5]. In [Mon22], S. Monavari described a *canonical signs* compatible to the above choice of square roots as follows: for any solid partition π,

$$o(\mathcal{L})|_{Z_\pi} := |\pi| + \#\{(i,j,k,l) \in \pi : i = j = k < l\};$$

and for any N-colored solid partition π,

$$o(\mathcal{L})|_{Z_\pi} := \sum_{i=1}^{N} o(\mathcal{L})|_{Z_i}.$$

This is the choice of signs that we used in our SageMath program when checking our conjectures.

5.3. **Factor of $c_3(X)$.** In the surface case, we saw the powers in the universal series of virtual invariants are multiples of $c_1(S)$. In the $X = \mathbb{C}^4$ case, we shall show that if the universal expressions exist, then they are multiples of $c_3(X)$ by showing show that

$$e\left(-\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}\right)$$

has $c_3(X) = -(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)$ in its numerator. This factor of $c_3(X)$ and the weak Segre-Verlinde correspondence in the surface case shall motivate Conjecture 1.16.

It suffices to show that this term vanishes when we set $\lambda_i = -\lambda_j$ for $i \neq j$ in $\{1,2,3\}$. By symmetry, we may assume $i = 1, j = 2$. Recall e is the top equivariant Chern class, which vanishes when $\lambda_1 = -\lambda_2$ if $-\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}$ has a T-fixed summand when $t_1 = t_2^{-1}$, i.e. the character of $\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}$ in $K_{\mathsf{T}}(\mathrm{pt})$ having a strictly negative constant term. This occurs if and only if the image of $T^{\mathrm{vir}}_{Z_\pi}$ in

$$\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(t_1t_2 - 1, t_3t_4 - 1)$$

has a strictly negative constant term (which is necessarily a negative even integer). From (5.3), we see it suffices to show this for the term

$$Q_\pi + t_1t_2t_3t_4\overline{Q_\pi} - t_1t_2t_3t_4P_{1234}\overline{Q_\pi}Q_\pi$$

whenever π is a non-trivial solid partition.

Lemma 5.6. *For any non-trivial solid partition π, the expression*

$$Q_\pi + t_1t_2t_3t_4\overline{Q_\pi} - t_1t_2t_3t_4P_{1234}\overline{Q_\pi}Q_\pi$$

has a strictly negative constant term when viewed in the quotient ring

$$\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(t_1t_2 - 1, t_3t_4 - 1).$$

Proof. Let $x = t_1 = \frac{1}{t_2}$, $y = t_3 = \frac{1}{t_4}$, so that

$$\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(t_1t_2 - 1, t_3t_4 - 1) = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}].$$

Let P_π be the image of Q_π in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, then

$$T^{\mathrm{vir}}|_{Z_\pi} = P_\pi + \overline{P_\pi} - P_\pi\overline{P_\pi}(1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y}). \tag{5.4}$$

Write

$$P_\pi = \sum_{i,j\in\mathbb{Z}} p_{i,j}x^iy^j.$$

The image of $\overline{Q_\pi}$ is then

$$\overline{P_\pi} = \sum_{i,j\in\mathbb{Z}} p_{i,j} x^{-i} y^{-j}.$$

We see the constant terms of P_π and $\overline{P_\pi}$ are both $p_{0,0}$. By definition, all monomial terms in Q_π have positive coefficients, and Q_π has constant term 1, so $p_{0,0} > 0$. We need to find the constant term of $P_\pi \overline{P_\pi}(1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y})$.

Observe that

$$(1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y}) = 4 - 2\left(x + y + \frac{1}{x} + \frac{1}{y}\right) + \left(xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x}\right).$$

Write $F = \sum f_{i,j} x^i y^j$, the constant term of $F \cdot (1-x)(1-\frac{1}{x})(1-y)(1-\frac{1}{y})$ is equal to

$$4f_{0,0} - 2(f_{0,1} + f_{1,0} + f_{0,-1} + f_{-1,0}) + (f_{1,1} + f_{1,-1} + f_{-1,1} + f_{-1,-1}). \tag{5.5}$$

If we set $F = P_\pi \overline{P_\pi}$, then

$$f_{i,j} = \sum_{\substack{a-c=i\\ b-d=j}} p_{a,b} p_{c,d}.$$

In particular,

$$\begin{aligned}
f_{0,0} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}^2,\\
f_{0,1} + f_{1,0} + f_{0,-1} + f_{-1,0} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}(p_{a-1,b} + p_{a+1,b} + p_{a,b-1} + p_{a,b+1}),\\
f_{1,1} + f_{1,-1} + f_{-1,1} + f_{-1,-1} &= \sum_{a,b\in\mathbb{Z}} p_{a,b}(p_{a+1,b+1} + p_{a+1,b-1} + p_{a-1,b-1} + p_{a-1,b+1}).
\end{aligned}$$

Denote

$$\begin{aligned}
s_{a,b} &= 4p_{a,b} - 2(p_{a-1,b} + p_{a+1,b} + p_{a,b-1} + p_{a,b+1});\\
&\quad + (p_{a+1,b+1} + p_{a+1,b-1} + p_{a-1,b-1} + p_{a-1,b+1}),\\
s^{++}_{a,b} &= p_{a,b} - (p_{a+1,b} + p_{a,b+1}) + p_{a+1,b+1},\\
s^{+-}_{a,b} &= p_{a,b} - (p_{a+1,b} + p_{a,b-1}) + p_{a+1,b-1},\\
s^{-+}_{a,b} &= p_{a,b} - (p_{a-1,b} + p_{a,b+1}) + p_{a-1,b+1},\\
s^{--}_{a,b} &= p_{a,b} - (p_{a-1,b} + p_{a,b-1}) + p_{a-1,b-1};
\end{aligned}$$

$$\begin{aligned}
S^{++} &= \sum_{a,b\geq 0} p_{a,b} s^{++}_{a,b} + p_{a+1,b} s^{-+}_{a+1,b} + p_{a,b+1} s^{+-}_{a,b+1} + p_{a+1,b+1} s^{--}_{a+1,b+1},\\
S^{+-} &= \sum_{a\geq 0, b\leq 0} p_{a,b} s^{+-}_{a,b} + p_{a+1,b} s^{--}_{a+1,b} + p_{a,b-1} s^{++}_{a,b-1} + p_{a+1,b-1} s^{-+}_{a+1,b-1},\\
S^{-+} &= \sum_{a\leq 0, b\geq 0} p_{a,b} s^{-+}_{a,b} + p_{a+1,b} s^{++}_{a+1,b} + p_{a,b+1} s^{--}_{a,b+1} + p_{a+1,b+1} s^{+-}_{a+1,b+1},\\
S^{--} &= \sum_{a,b\leq 0} p_{a,b} s^{--}_{a,b} + p_{a+1,b} s^{+-}_{a+1,b} + p_{a,b+1} s^{-+}_{a,b+1} + p_{a+1,b+1} s^{++}_{a+1,b+1}.
\end{aligned}$$

Then (5.5) becomes

$$\begin{aligned}\sum_{a,b\in\mathbb{Z}} p_{a,b}s_{a,b} = &\sum_{a,b\in\mathbb{Z}} p_{a,b}\cdot(s_{a,b}^{++}+s_{a,b}^{+-}+s_{a,b}^{-+}+s_{a,b}^{--})\\ =&S^{++}+S^{+-}+S^{-+}+S^{--}.\end{aligned}$$

For the remainder of this proof, we shall show $S^{++}\geq p_{0,0}$. The same will hold for the summands S^{+-}, S^{-+}, S^{--} by symmetry. We conclude the value of (5.5) is at least $4p_{0,0}$. Hence by (5.4) the constant term of $T^{\mathrm{vir}}|_{Z_\pi}$ is at most $-2p_{0,0}<0$, and we are done.

Recall

$$Q_\pi = \sum_{(i,j,k,l)\in\pi} t_1^i t_2^j t_3^k t_4^l,$$

so

$$P_\pi = \sum_{(i,j,k,l)\in\pi} x^{i-j}y^{k-l}, \text{ and}$$

$$p_{a,b} = \#\{(i,j,k,l)\in\pi : i-j=a, k-l=b\}.$$

Fix k and l, then the set $\{(i,j):(i,j,k,l)\in\pi\}$ is a plane partition. By property (2.2) of solid partitions, for fixed $b=k-l$, we have $p_{a,b}\geq p_{a+1,b}$ when $a\geq 0$. For the same reason, we have $p_{a,b}\geq p_{a,b+1}$ when $b\geq 0$. Therefore the numbers $(p_{a,b})_{a,b\geq 0}$ are non-increasing as the pair (a,b) move away from the origin.

We apply induction on $\max\{a : p_{a,0}\neq 0\}$. Suppose for all sequences $(q_{a,b})$ with $\max\{a : q_{a,0}\neq 0\} < \max\{a : p_{a,0}\neq 0\}$, we have

$$S^{++}(q_{a,b})\geq q_{0,0}$$

whenever the sequence $(q_{a,b})_{a,b\geq 0}$ satisfies $q_{a,b}$ is non-increasing in a,b. The base case is simply when $q_{a,b}=0$ for all a,b, which sums to 0. Let $q_{a,b}=p_{a+1,b}$, then

$$\begin{aligned}S^{++}(p_{a,b}) = &\sum_{a,b\geq 0}(p_{a,b}-p_{a+1,b}-p_{a,b+1}+p_{a+1,b+1})^2\\ =&S^{++}(q_{a,b})+\sum_{b\geq 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2\\ \geq&p_{1,0}+\sum_{b\geq 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2\end{aligned}\tag{5.6}$$

where the first equality follows from the definition and the inequality is by induction hypothesis.

Now apply another induction on the value of $\max\{b : p_{0,b}\neq 0\}$. The induction hypothesis is that for any sequences $(q_{a,b})$ with $q_{a,b}$ non-increasing in a,b and $\max\{b : q_{0,b}\neq 0\}<\max\{b : p_{0,b}\neq 0\}$, we have

$$\sum_{b\geq 0}(q_{0,b}-q_{1,b}-q_{0,b+1}+q_{1,b+1})^2\geq q_{0,0}-q_{1,0}.$$

Again, the base case is trivial, and we can apply the hypothesis to $q_{a,b}=p_{a,b+1}$, giving us

$$\sum_{b\geq 1}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2\geq p_{0,1}-p_{1,1}$$

So we have the following inequalities

$$\begin{aligned}&(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1})^2+\sum_{b\geq 1}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2-(p_{0,0}-p_{1,0})\\ \geq&(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1})^2-(p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1})\\ \geq&0\end{aligned}$$

where the last inequality is due to $p_{0,0}-p_{1,0}-p_{0,1}+p_{1,1}$ being an integer. Therefore

$$\sum_{b\geq 0}(p_{0,b}-p_{1,b}-p_{0,b+1}+p_{1,b+1})^2 \geq p_{0,0}-p_{1,0},$$

finishing the second induction. By (5.6),

$$S^{++}(p_{a,b}) \geq p_{1,0}+(p_{0,0}-p_{1,0}) = p_{0,0},$$

which finishes the first induction and the proof.

□

5.4. **Cohomological limits.** Recall the proof of Theorem 3.9 mainly involved showing that the genus I_S is admissible in the sense of Definition 2.6. Also, by Proposition 2.7, universal series expressions for the Nekrasov genus, and therefore the Segre and Verlinde series, can be obtained if and only if the Nekrasov genus is admissible. Thus one might ask when the Nekrasov genus is admissible. For rank $r=N$ case, we shall show that admissibility is a consequence of the following explicit formula conjectured by Nekrasov-Piazzalunga [NP19, Section 2.5]. Denote

$$[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}.$$

Conjecture 5.7 (Nekrasov-Piazzalunga)**.** *There exists some choice of signs $o(\mathcal{L})$ such that for $E=\oplus_{i=1}^N \mathcal{O}_X\langle y_i\rangle$, $V=\oplus_{i=1}^N \mathcal{O}_X\langle v_i\rangle$,*

$$\mathcal{N}_X(E,V;q) = \mathrm{Exp}\left(\frac{[t_1t_2][t_2t_3][t_1t_3]}{[t_1][t_2][t_3][t_4]}\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}\right)$$

with a change of variable $s=\prod_{i=1}^N y_iv_i$.

Proposition 5.8. *Nekrasov-Piazzalunga's Conjecture 5.7 implies Nekrasov's genus of N_X for rank $r=N$ is admissible with respect to the variables t_1,t_2,t_3,t_4.*

Proof. Expand the term inside the plethystic exponential, specializing with the relation $t_1t_2t_3t_4=1$, we have

$$\frac{[t_1t_2][t_2t_3][t_1t_3][s]}{[t_1][t_2][t_3][t_4][s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]} = \frac{(1-t_1t_2)(1-t_2t_3)(1-t_1t_3)}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)}\cdot\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}$$

Recall Definition 2.6, we have

$$L=(1-t_1t_2)(1-t_2t_3)(1-t_1t_3)\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}$$

is a series in $q, y_1^{\pm\frac{1}{2}},\dots,y_N^{\pm\frac{1}{2}}, v_1^{\pm\frac{1}{2}},\dots,v_r^{\pm\frac{1}{2}}$ whose coefficients are polynomials in t_1,t_2,t_3,t_4, as required. □

Lastly, we prove the claim made in the introduction that Conjecture 1.17 is a consequence of Conjecture 5.7 in the $X=\mathbb{C}^4$ case.

Proposition 5.9. *Let $X=\mathbb{C}^4$. If Conjecture 5.7 holds for some choice of signs, then Conjecture 1.17 holds for $Y=X$ for some choice of signs.*

In addition, Conjecture 5.7 implies the following Donaldson-Thomas integral:

$$\sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_{\mathbb{C}^4}(E,n)]^{vir}_{o(\mathcal{L})}} 1 := \sum_{n=0}^{\infty} q^n \sum_{Z\in \mathrm{Quot}_X(E,n)^{\mathsf{T}}} (-1)^{o(\mathcal{L})|_Z}\frac{1}{e_{\mathsf{T}}\left(\sqrt{T_Z^{vir}}\right)}$$
$$= \begin{cases} e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}q}, & \textit{when } N=1 \\ 1, & \textit{otherwise.}\end{cases}$$

Remark 5.10. One can compare this to the 3-fold case where [FMR21, Theorem 7.2] states

$$\sum_{n=0}^{\infty} q^n \int_{[\mathrm{Quot}_{\mathbb{C}^3}(E,n)]^{\mathrm{vir}}} 1 = M((-1)^N q)^{-N\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3}}.$$

Here M denotes the MacMahon function.

Proof. We shall compute the following limit using both the definition and the expression from Conjecture 5.7, then compare the two sides:

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E, V; \frac{Q}{w_N}\right)\bigg|_{\lambda_i \rightsquigarrow \varepsilon\lambda_i, m_i\rightsquigarrow \varepsilon(1+m_i), w_i\rightsquigarrow \varepsilon w_i}.$$

Let $V = \oplus_{i=1}^N \mathcal{O}_X\langle v_i\rangle$ be a rank N bundle, then for any $Z_\pi \in \mathrm{Quot}_X(E,n)^{\mathsf{T}}$, we have

$$\begin{aligned}
&\frac{\mathrm{ch}_{\mathsf{T}}(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}})}{\mathrm{ch}_{\mathsf{T}}(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^{\vee})}\mathrm{ch}_{\mathsf{T}}\left(\frac{\Lambda_{-1}}{\det^{\frac{1}{2}}}V^{[n]}|_{Z_\pi}\right)\bigg|_{\lambda_i \rightsquigarrow \varepsilon\lambda_i, m_i\rightsquigarrow \varepsilon(1+m_i), w_i\rightsquigarrow \varepsilon w_i}\\
=&\varepsilon^{Nn-Nn}\frac{e_{\mathsf{T}}(V^{[n]}|_{Z_\pi}) + O(\varepsilon)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right) + O(\varepsilon)}\\
=&\frac{\prod_{i=1}^N\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j+a\lambda_1+b\lambda_2+c\lambda_3+d\lambda_4)+O(\varepsilon)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)+O(\varepsilon)}.
\end{aligned}$$

Take limit $\varepsilon \to 0$ and let $Q = m_N q$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\mathrm{ch}_{\mathsf{T}}(\sqrt{K^{\mathrm{vir}}|_{Z_\pi}}^{\frac{1}{2}})}{\mathrm{ch}_{\mathsf{T}}(\Lambda_{-1}\sqrt{T^{\mathrm{vir}}|_{Z_\pi}}^{\vee})}\mathrm{ch}_{\mathsf{T}}\left(\frac{\Lambda_{-1}}{\det^{\frac{1}{2}}}V^{[n]}|_{Z_\pi}\right)\bigg|_{\lambda_i \rightsquigarrow \varepsilon\lambda_i, m_i\rightsquigarrow \varepsilon(1+m_i), w_i\rightsquigarrow \varepsilon w_i}\cdot q^n\\
=&\frac{\prod_{i=1}^N\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j-a\lambda_1-b\lambda_2-c\lambda_3-d\lambda_4)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)}\cdot\frac{Q^n}{m_N^n}\\
=&\frac{\prod_{i=1}^{N-1}\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}(1+w_i+m_j-a\lambda_1-b\lambda_2-c\lambda_3-d\lambda_4)}{e_{\mathsf{T}}\left(\sqrt{T^{\mathrm{vir}}_{Z_\pi}}\right)}\\
&\cdot\prod_{j=1}^N\prod_{(a,b,c,d)\in\pi^{(j)}}\left(1+\frac{m_j}{w_N}-\frac{a\lambda_1}{w_N}-\frac{b\lambda_2}{w_N}-\frac{c\lambda_3}{w_N}-\frac{d\lambda_4}{w_N}\right)Q^n.
\end{aligned}$$

Now take $w_N \to \infty$ and substitute into Definition 5.2. Let $V' = \oplus_{i=1}^{N-1}\mathcal{O}_X\langle v_i\rangle$, then

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E, V; \frac{Q}{w_N}\right)\bigg|_{\lambda_i \rightsquigarrow \varepsilon\lambda_i, m_i\rightsquigarrow \varepsilon(1+m_i), w_i\rightsquigarrow \varepsilon w_i} = \mathcal{C}_X(E, V'; Q). \tag{5.7}$$

On the other hand, we apply the same procedure to

$$\mathcal{N}_X(E,V;q) = \mathrm{Exp}\left(\frac{[t_1t_2][t_2t_3][t_1t_3]}{[t_1][t_2][t_3][t_4]}\frac{[s]}{[s^{\frac{1}{2}}q][s^{-\frac{1}{2}}q]}\right).$$

For $n \geq 1$, we have

$$\begin{aligned}
&\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \frac{[t_1^n t_2^n][t_2^n t_3^n][t_1^n t_3^n]}{[t_1^m][t_2^n][t_3^n][t_4^n]} \frac{[\prod y_i^n v_i^n]}{[\prod y_i^{\frac{n}{2}} v_i^{\frac{n}{2}} q^n][\prod y_i^{-\frac{n}{2}} v_i^{-\frac{n}{2}} q^n]}\Big|_{\lambda_i\rightsquigarrow\varepsilon\lambda_i, m_i\rightsquigarrow\varepsilon(1+m_i), w_i\rightsquigarrow\varepsilon w_i}\\
=&\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \frac{(\varepsilon n)^3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)+O(\varepsilon^5)}{(\varepsilon n)^4\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)+O(\varepsilon^5)} \cdot \frac{(\varepsilon n)\sum_i(1+m_i+w_i)+O(\varepsilon)}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})^2}\\
=&\lim_{w_N\to\infty} \frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)} \cdot \frac{\sum_i(1+m_i+w_i)(\frac{Q}{w_N})^n}{(1-(\frac{Q}{w_N})^n)^2}\\
=&\begin{cases} \frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q, & \text{when } n=1\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Together with (5.7), we have

$$\begin{aligned}
\mathcal{C}_X(E,V';Q) =& e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q}\\
=& \exp\left(Q\int_X c_3(X)\right).
\end{aligned}$$

This is exactly Conjecture 1.17.

With the same method, we can take limits

$$\lim_{\substack{\varepsilon\to 0\\ w_N\to\infty}} \mathcal{N}_X\left(E, V; \frac{Q}{w_N w_{N-1}\dots w_{N-i+1}}\right)$$

for $1 < i \leq N$ and V of rank $N-i$, and get

$$\mathcal{C}_X(E,V;Q) = 1.$$

In particular, when $i = N$ and $N > 1$, we have

$$\sum_{n=0}^{\infty} Q^n \int_{[\mathrm{Quot}_{\mathbb{C}^4}(E,n)]^{\mathrm{vir}}_{o(\mathcal{L})}} 1 = 1.$$

$\square$

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